

Partially quenched chiral perturbation theory in the ϵ -regime

Poul H. Damgaard ^a

^a*The Niels Bohr Institute, The Niels Bohr International Academy, Blegdamsvej 17
DK-2100 Copenhagen Ø Denmark*

Hidenori Fukaya ^b

^b*Theoretical Physics Laboratory, RIKEN, Wako 351-0198, Japan*

Abstract

We calculate meson correlators in the ϵ -regime within partially quenched chiral perturbation theory. The valence quark masses and sea quark masses can be chosen arbitrary and all non-degenerate. Taking some of the sea quark masses to infinity, one obtains a smooth connection among the theories with different number of flavors, as well as the quenched theory. These results can be directly compared with lattice QCD simulations.

1 Introduction

In the low energy limit, the dynamics of QCD is described by the pion fields which appear as pseudo-Nambu-Goldstone bosons accompanying the spontaneous breaking of chiral symmetry. Chiral dynamics and chiral perturbation theory (ChPT), play an essential role in understanding the interactions among the pions themselves, as well as their couplings with the other (heavier) hadrons and sources.

The fundamental parameters of ChPT are unknown coupling constants in the effective theory. But they can be determined by non-perturbative and first-principle calculations in the underlying theory, QCD. The most important low-energy constants are the chiral condensate Σ , and the pion decay constant, F ,

Email addresses: phdamg@nbi.dk (Poul H. Damgaard), hfukaya@riken.jp (Hidenori Fukaya).

both at leading order. Numerical simulations of lattice QCD give currently the most promising approach to achieving this non-perturbative determination. Because such computer simulations necessarily are restricted to finite volumes, it is important to understand finite-volume effects in the effective field theory.

Near the chiral limit, the finite-volume effects become increasingly significant due to the diverging correlation length of the Goldstone bosons. In particular, when the pion correlation length, or the inverse of the pion mass m_π overcomes the size of the box L ,

$$\frac{1}{\Lambda_{\text{QCD}}} \ll L \ll \frac{1}{m_\pi}, \quad (1.1)$$

where Λ_{QCD} is the QCD scale, the zero-momentum mode has to be treated non-perturbatively and the ChPT has to be performed in a way that achieves this in a systematic fashion: the so-called ϵ -expansion [1,2,3,4]. New counting rules are needed to order the perturbative expansion in this case. In units of the ultraviolet cut-off, the expansion parameter can be defined as ϵ , where, unusually, the pion mass m_π is not treated as being of the same order as pion momentum p . Instead,

$$m_\pi \sim p^2 \sim \epsilon^2, \quad (1.2)$$

while the inverse of space-time volume V and quark mass m_q are being of $\mathcal{O}(\epsilon^4)$. The particularly important combination $m_q \Sigma V$ is thus treated as of order unity. With this expansion, the precise analytical predictions for physical observables in the low-energy sector of QCD at finite volume V are expressed in terms of the low-energy constants at infinite volume. By comparing numerical results at finite volume with these predictions, one can thus extract the infinite-volume constants directly from finite-volume simulations, without the need for extrapolations of data to infinite volume. The closer one gets to the chiral limit, the bigger is the advantage of this approach.

A few years ago, the predictions for correlation functions of ChPT were extended to the cases of both quenched QCD and full QCD at sectors of fixed gauge-field topology [5,6]. For the chiral condensate, the studies were also extended to the partially quenched cases [7,8]. In quenched QCD these analytical predictions suffer in the ϵ -regime from quenched finite-volume logarithms [9]. Strictly speaking, such logarithms prevent taking the infinite-volume limit, and basically invalidate the whole chiral expansion in this regime for the quenched theory. The hope is that in finite ranges of volume, the resulting predictions may still have a certain range of validity. For both quenched and unquenched cases these computations were restricted to the case of degenerate, light, quarks.

As mentioned above, the great advantage of the predictions for correlators in the ϵ -regime is that they appear almost tailored for numerical lattice computations near the chiral limit. Indeed, exploratory studies of these correlation functions have already shown the great potential [10,11,12,13,14,15,16]. In par-

ticular, the possibility of using to one's advantage the role played by fixing topology in finite volume has been clearly demonstrated. Also the analytic handle one has on the quark mass dependence due to the finite size effects in the ϵ -regime has proved helpful in reducing the systematic errors of lattice simulations.

In this paper, we generalize these analytical computations of correlation functions in the ϵ -regime to the partially quenched theory with separate valence and sea quarks, both of which are taken to be non-degenerate. The chiral condensate and pseudo-scalar and scalar meson correlators are calculated as functions of non-degenerate quark masses, topological charge, and the volume of the Euclidean space-time V . In a separate forthcoming publication [17] we will present the analogous results for the vector and axial vector channels.

In practice, the ϵ -regime is not trivially reached in numerical simulations. It is therefore important to be able to go close to the chiral limit, but still only marginally in the ϵ -regime, while valence quark masses are taken to that regime. In fact, although our aim in this paper is the ϵ -regime, our partially quenched chiral perturbation theory (PQChPT) [18,19] in the ϵ -regime smoothly connects all the theories with a different number of flavors as a function of the sea quark masses. In this way, our calculation interpolates between the ϵ -regime and the more conventional p -regime, and in one kinematical regime also mixes the two expansions. For this reason we provide expressions not just with the (simpler) ϵ -expansion propagators, but the more general expressions. If one of the sea quark masses is taken to infinity in the N_f -flavor theory, it converges to the $(N_f - 1)$ -flavor theory. Even the quenched theory can be obtained by carefully introducing the flavor singlet field before taking all the sea quark masses to infinity. Of course, the low-energy constants in addition have an inherent flavor dependence that is beyond our control.

Our results have wide applicability to unquenched lattice QCD studies near the chiral limit [20,21,22]. One can choose various valence quark masses with a fixed sea quark mass. The partially quenched condensate and meson correlators can be compared with simulations of heavier quarks which are perhaps only marginally in the ϵ -regime (or beyond) while the valence quark mass is still in the ϵ -regime.

This paper is organized as follows. In Sec.2, we describe the leading contribution of the partition function of PQChPT in the ϵ -expansion. We discuss, in particular, the exact non-perturbative integral of the zero-modes [23,24] which plays a crucial role in deriving both the chiral condensate and meson correlators in this extended theory. As one fundamental building block of this work, the chiral condensate and its 1-loop level correction are obtained in Sec.3. The exact zero-mode integrals in the replica limit are calculated in Sec.4. We also derive a non-trivial identity which follows from the unitarity of the group

integrals. In Sec.5, our main results on meson correlators are presented. We plot $N_f = 2$ connected pseudo-scalar and scalar correlators as examples. The conclusions are given in Sec.6.

2 The partition function of PQChPT

Our starting point is the $(N_f + N)$ -flavor chiral Lagrangian,

$$\begin{aligned} \mathcal{L} = & \frac{F^2}{4} \text{Tr}(\partial_\mu U(x)^\dagger \partial_\mu U(x)) - \frac{\Sigma}{2} \text{Tr}(\mathcal{M}^\dagger U(x) + \mathcal{M} U(x)^\dagger) \\ & + \frac{m_0^2}{2N_c} \Phi_0^2(x) + \frac{\alpha}{2N_c} \partial_\mu \Phi_0(x) \partial_\mu \Phi_0(x), \end{aligned} \quad (2.1)$$

where Σ and F are the chiral condensate and the pion decay constant at infinite volume, both in the chiral limit. In the mass matrix

$$\mathcal{M} = \text{diag}(\underbrace{m_1, m_2, \dots, m_{N_f}}_{N_f}, \underbrace{m_v, \dots, m_v}_N), \quad (2.2)$$

we have in mind a situation in which the valence quark mass is always taken in the ϵ -regime of $m_v \Sigma V \sim \mathcal{O}(1)$, while the physical sea quark mass m_i may vary more freely. Unlike standard chiral perturbation theory, $U(x)$ is an element of the $U(N_f + N)$ group, and the flavor-singlet field, $\Phi_0(x) \equiv \frac{-iF}{\sqrt{2}} \text{Tr} \ln U(x)$, is introduced explicitly as a physical degree of freedom with additional constants, m_0 and α . The number of colors is denoted by N_c . As is well known, in this partially quenched theory one can normally take the $m_0 \rightarrow \infty$ limit without difficulty. In terms of first replicated and then quenched valence quarks one is then going from $U(N_f + N)$ to $SU(N_f)$ in a smooth way. Then Φ_0 can be decoupled from the theory. Of course, trouble arises again if we consider the theory in a regime where the sea quark masses m_i have effectively decoupled. We will discuss this issue below.

Separating the zero-mode, U_0 , and the non-zero modes, $\xi(x)$,

$$U(x) = U_0 \exp(i\sqrt{2}\xi(x)/F), \quad (2.3)$$

we consider three types of expansion of the partition function in a sector of fixed topological charge ν :

- (1) Both of the valence and sea quarks are in the ϵ -regime:

$$Z_{N_f+N}^\nu(m_v, \{m_i\}) = \int_{U(N_f+N)} dU_0 d\xi \det U_0^\nu \exp \left[\frac{\Sigma V}{2} \text{Tr}[\mathcal{M}^\dagger U_0 + \mathcal{M} U_0^\dagger] \right. \\ \left. + \int d^4x \left(-\frac{1}{2} \text{Tr}[\partial_\mu \xi \partial_\mu \xi] \right) + \dots \right], \quad (2.4)$$

where the $m_0 \rightarrow \infty$ limit is taken and the singlet Φ_0 is decoupled from the theory.

- (2) The sea quarks are in the p -regime (but light enough that we can still disregard effects of the singlet field):

$$Z_{N_f+N}^\nu(m_v, \{m_i\}) = \int_{U(N_f+N)} dU_0 d\xi \det U_0^\nu \exp \left[\frac{\Sigma V}{2} \text{Tr}[\mathcal{M}^\dagger U_0 + \mathcal{M} U_0^\dagger] \right. \\ \left. + \int d^4x \left(-\frac{1}{2} \text{Tr}[\partial_\mu \xi \partial_\mu \xi] - \left(\frac{\Sigma}{F^2} \right) \text{Tr}[\mathcal{M} \xi^2] + \dots \right) \right]. \quad (2.5)$$

Here the valence sector is expanded as in ϵ -expansion, while the sea sector is considered in the p -regime. Note that the mass term in the valence sector, $m_v \Sigma \xi^2 / F^2$, is of $\mathcal{O}(\epsilon^4)$ but not ignored here, in order to see a smooth transition to the p -regime.

- (3) The sea quarks are heavy:

$$Z_{N_f+N}^\nu(m_v, \{m_i\}) = \int_{U(N_f+N)} dU_0 d\xi \det U_0^\nu \exp \left[\frac{\Sigma V}{2} \text{Tr}[\mathcal{M}^\dagger U_0 + \mathcal{M} U_0^\dagger] \right. \\ \left. + \int d^4x \left(-\frac{1}{2} \text{Tr}[\partial_\mu \xi \partial_\mu \xi] - \left(\frac{\Sigma}{F^2} \right) \text{Tr}[\mathcal{M} \xi^2] \right. \right. \\ \left. \left. - \frac{m_0^2}{2N_c} (\text{Tr} \xi)^2 - \frac{\alpha}{2N_c} (\partial_\mu \text{Tr} \xi)^2 \right) + \dots \right]. \quad (2.6)$$

As the sea quark mass increases, new terms (the non-zero mode's mass term, and the singlet fields) come in. Thus, Eq.(2.6) is the most general form.

In the following, we implicitly restrict ourselves to sectors of small enough fixed topology ν for the expansion to be valid [25]. Indeed, this is an essential assumption so that in Eq.(2.5) and Eq.(2.6) one needs only the mass term $\text{Tr}[\mathcal{M} \xi^2]$ instead of $\text{Tr}[\mathcal{M}(U_0 + U_0^\dagger) \xi^2] / 2$ ¹.

For the zero-mode integrals, one needs exact formulae of the group integrals over $U(N_f + N)$ and means of taking the replica limit. As described in detail in ref. [5], if we wish to consider correlation functions with N_v external valence quarks we must embed the N_v valence quarks in a theory with N replicated quarks in total (of which $N - N_v$ do not couple to the external sources), and then take the limit $N \rightarrow 0$. Alternatively, one can consider a theory with N_v additional bosonic flavors of common mass m_v . It is easy to understand in the quark determinant picture that this limit is equivalent to the replica limit:

¹ We thank P. Hernandez for stressing this point.

$$\begin{aligned} \lim_{N \rightarrow 0} \det(D + m)^{N_f} \left(\prod_i^{N_v} \det(D + m_v + J_i) \right) \det(D + m_v)^{N - N_v} \\ = \det(D + m)^{N_f} \frac{\prod_i^{N_v} \det(D + m_v + J_i)}{\det(D + m_v)^{N_v}}. \end{aligned} \quad (2.7)$$

Here the left hand side is the replica prescription, while the right hand side is the prescription of the graded formalism.

The zero-mode partition function of n bosons and m fermions are analytically known [23,24],

$$\mathcal{Z}_{n,m}^\nu(\{\mu_i\}) = \frac{\det[\mu_i^{j-1} \mathcal{J}_{\nu+j-1}(\mu_i)]_{i,j=1,\dots,n+m}}{\prod_{j>i=1}^n (\mu_j^2 - \mu_i^2) \prod_{j>i=n+1}^{n+m} (\mu_j^2 - \mu_i^2)}, \quad (2.8)$$

where $\mu_i = m_i \Sigma V$. Here \mathcal{J} 's are defined as $\mathcal{J}_{\nu+j-1}(\mu_i) \equiv (-1)^{j-1} K_{\nu+j-1}(\mu_i)$ for $i = 1, \dots, n$ and $\mathcal{J}_{\nu+j-1}(\mu_i) \equiv I_{\nu+j-1}(\mu_i)$ for $i = n+1, \dots, n+m$, where I_ν and K_ν are the modified Bessel functions. Of particular importance is the case $(n, m) = (1, N_f + 1)$:

$$\begin{aligned} \mathcal{Z}_{1,1+N_f}^\nu(x|y, \{z_i\}) = \frac{1}{\prod_{i=1}^{N_f} (z_i^2 - y^2) \prod_{k>j}^{N_f} (z_k^2 - z_j^2)} \\ \times \det \begin{pmatrix} K_\nu(x) & I_\nu(y) & I_\nu(z_1) & I_\nu(z_2) & \cdots \\ -x K_{\nu+1}(x) & y I_{\nu+1}(y) & z_1 I_{\nu+1}(z_1) & z_2 I_{\nu+1}(z_2) & \cdots \\ x^2 K_{\nu+2}(x) & y^2 I_{\nu+2}(y) & z_1^2 I_{\nu+2}(z_1) & z_2^2 I_{\nu+2}(z_2) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \end{aligned} \quad (2.9)$$

where $x = m_b \Sigma V$ (m_b denotes the bosonic quark mass), $y = m_v \Sigma V$ and $z_i = m_i \Sigma V$. One notes that

$$\lim_{x \rightarrow y} \mathcal{Z}_{1,1+N_f}^\nu(x|y, \{z_i\}) = \mathcal{Z}_{0,N_f}^\nu(\{z_i\}), \quad (2.10)$$

and therefore,

$$-\lim_{x \rightarrow y} \partial_x \mathcal{Z}_{1,1+N_f}^\nu(x|y, \{z_i\}) = \lim_{x \rightarrow y} \partial_y \mathcal{Z}_{1,1+N_f}^\nu(x|y, \{z_i\}). \quad (2.11)$$

It is also remarkable that

$$\begin{aligned} \mathcal{Z}_{1,1+N_f}^\nu(x|y, \{z_1, z_2, \dots, z_{j-1}, z_j \rightarrow \infty, z_{j+1}, \dots, z_{N_f}\}) = \\ \mathcal{Z}_{1,1+(N_f-1)}^\nu(x|y, \{z_1, z_2, \dots, z_{j-1}, z_{j+1}, \dots, z_{N_f}\}) \\ \times \left(\left[z_j^{1-N_f} \mathcal{Z}_{0,1}^{\nu+N_f+1}(z_j) \right] \Big|_{z_j \rightarrow \infty} \right), \end{aligned} \quad (2.12)$$

which is consistent with the intuitive notion that the $N_f - 1$ -flavor theory can be obtained in the limit of large sea-quark mass, $m_j \rightarrow \infty$, up to a normalization factor $\left[z_j^{1-N_f} \mathcal{Z}_{0,1}^{\nu+N_f+1}(z_j) \right] \Big|_{z_j \rightarrow \infty}$. This decoupling is of course more general. For example, in the case of $\{z_i \rightarrow \infty \text{ (for all } i)\}$ one obtains $\mathcal{Z}_{1,1}'$, the leading partition function of quenched chiral perturbation theory.

For different valence quarks (with non-degenerate valence masses), we will also need $\mathcal{Z}_{2,2+N_f}$:

$$\begin{aligned} \mathcal{Z}_{2,2+N_f}^\nu(x_1, x_2 | y_1, y_2, \{z_i\}) = & \frac{1}{(x_2^2 - x_1^2)(y_2^2 - y_1^2) \prod_{i=1}^{N_f} (z_i^2 - y_2^2)(z_i^2 - y_1^2) \prod_{k>j}^{N_f} (z_k^2 - z_j^2)} \\ & \times \det \begin{pmatrix} K_\nu(x_1) & K_\nu(x_2) & I_\nu(y_1) & I_\nu(y_2) & I_\nu(z_1) & \cdots \\ -x_1 K_{\nu+1}(x_1) & -x_2 K_{\nu+1}(x_2) & y_1 I_{\nu+1}(y_1) & y_2 I_{\nu+1}(y_2) & z_1 I_{\nu+1}(z_1) & \cdots \\ x_1^2 K_{\nu+2}(x_1) & x_2^2 K_{\nu+2}(x_2) & y_1^2 I_{\nu+2}(y_1) & y_2^2 I_{\nu+2}(y_2) & z_1^2 I_{\nu+2}(z_1) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \end{aligned} \quad (2.13)$$

Let us now define the propagator [26] of the fluctuation field ξ :

$$\begin{aligned} \bar{P}_{(ij)(kl)}(x-y) & \equiv \langle \xi_{ij}(x) \xi_{kl}(y) \rangle \\ & = \begin{cases} \delta_{il} \delta_{jk} \bar{\Delta}(M_{ij}^2 | x-y) & (i \neq j) \\ \delta_{ik} \delta_{jl} \bar{\Delta}(M_{ii}^2 | x-y) - \delta_{kl} \bar{G}(M_{ii}^2, M_{kk}^2 | x-y) & (i = j) \end{cases}, \end{aligned} \quad (2.14)$$

where the indices i, j, \dots can be taken both in the valence and sea sectors. Here $M_{ij}^2 = (m_i + m_j)\Sigma/F^2$ and

$$\bar{\Delta}(M_{ij}^2 | x) \equiv \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{p^2 + M_{ij}^2}, \quad (2.15)$$

$$\bar{G}(M_{ii}^2, M_{jj}^2 | x) \equiv \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx} (m_0^2 + \alpha p^2)/N_c}{(p^2 + M_{ii}^2)(p^2 + M_{jj}^2) \mathcal{F}(p^2)}, \quad (2.16)$$

$$\mathcal{F}(p^2) \equiv 1 + \sum_{f=1}^{N_f} \frac{(m_0^2 + \alpha p^2)/N_c}{p^2 + M_{ff}^2}. \quad (2.17)$$

Note that for small sea-quark masses, m_0 can be taken infinity,

$$\bar{G}(M_{ii}^2, M_{jj}^2|x) = \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{(p^2 + M_{ii}^2)(p^2 + M_{jj}^2) \left(\sum_f^{N_f} \frac{1}{p^2 + M_{ff}^2} \right)}. \quad (2.18)$$

Conversely, in the quenched limit of taking the sea quark masses to infinity, \bar{G} becomes

$$\bar{G}(M_{ii}^2, M_{jj}^2|x) \equiv \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}(m_0^2 + \alpha p^2)/N_c}{(p^2 + M_{ii}^2)(p^2 + M_{jj}^2)}. \quad (2.19)$$

3 The chiral condensate

At tree level, the partially quenched chiral condensate is obtained by the logarithmic x -derivative of the zero-mode partition function Eq.(2.9) followed by the $y \rightarrow x$ limit,

$$\begin{aligned} \frac{\Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} &= \lim_{N \rightarrow 0} \frac{1}{N} \langle \sum_v^N [U_0 + U_0^\dagger]_{vv} \rangle_{U_0} \\ &= - \lim_{y \rightarrow x} \frac{\partial}{\partial x} \ln \mathcal{Z}_{1,1+N_f}^\nu(x|y, \{z_i\}) \\ &= \frac{-1}{\mathcal{Z}_{0,N_f}^\nu(\{z_i\}) \prod_{i=1}^{N_f} (z_i^2 - x^2) \prod_{k>j}^{N_f} (z_k^2 - z_j^2)} \\ &\quad \times \det \begin{pmatrix} \partial_x K_\nu(x) & I_\nu(x) & I_\nu(z_1) & I_\nu(z_2) & \cdots \\ -\partial_x(x K_{\nu+1}(x)) & x I_{\nu+1}(x) & z_1 I_{\nu+1}(z_1) & z_2 I_{\nu+1}(z_2) & \cdots \\ \partial_x(x^2 K_{\nu+2}(x)) & x^2 I_{\nu+2}(x) & z_1^2 I_{\nu+2}(z_1) & z_2^2 I_{\nu+2}(z_2) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \end{aligned} \quad (3.1)$$

where $x = m_v \Sigma V$, $z_i = m_i \Sigma V$, and we use Eq.(2.10) for the denominator and the minus sign in the first line is due to the derivative with respect to the bosonic flavor. The quantity $\Sigma_\nu^{\text{PQ}}(x, \{z_i\})/\Sigma$ has two properties that follow immediately from the corresponding statements about the partition functions:

- (1) When one of the sea quark masses is large, the partially quenched chiral condensate reduces to that of the $(N_f - 1)$ -flavor theory;

$$\begin{aligned} \Sigma_\nu^{\text{PQ}}(x, \underbrace{\{z_1, \dots, z_j \rightarrow \infty, \dots, z_{N_f}\}}_{N_f \text{ flavors}}) / \Sigma = \\ \Sigma_\nu^{\text{PQ}}(x, \underbrace{\{z_1, \dots, z_{j-1}, z_{j+1} \dots z_{N_f}\}}_{(N_f-1) \text{ flavors}}) / \Sigma. \end{aligned} \quad (3.2)$$

As a particular case, one recovers the fully quenched condensate [5] when all sea quark masses are large, *i.e.* when $m_i \rightarrow \infty$ (See Fig. 1).

- (2) If the valence quark mass m_v is equal to one of sea quark masses m_j , it is equivalent to the j -th flavor quark condensate in the full theory:

$$\lim_{x \rightarrow z_j} \frac{\Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} = \partial_{z_j} \ln Z_{N_f}^\nu(\{z_i\}) \equiv \frac{\Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \quad (\text{for any } j). \quad (3.3)$$

The first property follows directly from Eq.(2.12) and the fact that

$$\mathcal{Z}_{0,1}^{\nu+N_f+1} / \mathcal{Z}_{0,1}^\nu \rightarrow 1 \quad (3.4)$$

in the large mass limit. The second property can be shown explicitly by using Eq.(2.10) and noting that the product $\prod_{i=1}^{N_f} (z_i^2 - x^2) \prod_{k>j}^{N_f} (z_k^2 - z_j^2)$ is always antisymmetric under a swap of $x \leftrightarrow z_j$. Together with the equation

$$\begin{aligned} \det \begin{pmatrix} K_\nu(x) & \partial_{z_j} I_\nu(z_j) & \dots & \overbrace{I_\nu(z_j)}^{j\text{-th column}} & \dots \\ -xK_{\nu+1}(x) & \partial_{z_j}(z_j I_{\nu+1}(z_j)) & \dots & z_j I_{\nu+1}(z_j) & \dots \\ x^2 K_{\nu+2}(x) & \partial_{z_j}(z_j^2 I_{\nu+2}(z_j)) & \dots & z_j^2 I_{\nu+2}(z_j) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} = \\ - \det \begin{pmatrix} K_\nu(x) & I_\nu(z_j) & \dots & \overbrace{\partial_{z_j} I_\nu(z_j)}^{j\text{-th column}} & \dots \\ -xK_{\nu+1}(x) & z_j I_{\nu+1}(z_j) & \dots & \partial_{z_j}(z_j I_{\nu+1}(z_j)) & \dots \\ x^2 K_{\nu+2}(x) & z_j^2 I_{\nu+2}(z_j) & \dots & \partial_{z_j}(z_j^2 I_{\nu+2}(z_j)) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \end{aligned} \quad (3.5)$$

which holds for any j , the statement follows.

For the normalization factor of the correlation functions to be computed in the following sections we need to evaluate the partition function to the given order in the expansion. This essentially boils down to an evaluation of the one-loop correction to the partially quenched chiral condensate. This correction can be calculated in a standard manner:

$$\begin{aligned}
\frac{\Sigma_{\nu}^{\text{PQ},1\text{-loop}}(x, \{z_i\})}{\Sigma} &= \frac{1}{\Sigma V} \lim_{N \rightarrow 0} \frac{1}{N} \frac{\partial}{\partial m_v} \ln Z_{N_f+N}^{\nu}(m_v, \{m_i\}) \\
&= \lim_{N \rightarrow 0} \left(1 - \frac{1}{F^2 N V} \sum_i^{N_f+N} \sum_v^N \left\langle \int d^4 x \xi_{vi}(x) \xi_{iv}(x) \right\rangle_{\xi} \right) \\
&\quad \times \frac{1}{N} \left\langle \sum_v^N [U_0 + U_0^{\dagger}]_{vv}, \right\rangle_{U_0}^{1\text{-loop}}, \tag{3.6}
\end{aligned}$$

where $\langle \dots \rangle_{\xi}$ denotes the integral over the ξ field, while the zero-mode integral $\langle \dots \rangle_{U_0}^{1\text{-loop}}$ can be calculated as above but due to the vacuum bubble, the arguments x and z_i are shifted, at 1-loop, as

$$m_v \Sigma V = x \rightarrow x^{\text{eff}} \quad \text{and} \quad m_i \Sigma V = z_i \rightarrow z_i^{\text{eff}}, \tag{3.7}$$

where

$$\begin{aligned}
\frac{x^{\text{eff}}}{x} &= \lim_{N \rightarrow 0} \left(1 - \frac{1}{F^2 N V} \sum_i^{N_f+N} \sum_v^N \left\langle \int d^4 x \xi_{iv} \xi_{vi} \right\rangle_{\xi} \right), \\
\frac{z_i^{\text{eff}}}{z_i} &= \lim_{N \rightarrow 0} \left(1 - \frac{1}{F^2 V} \sum_f^{N_f+N} \left\langle \int d^4 x \xi_{if} \xi_{fi} \right\rangle_{\xi} \right). \tag{3.8}
\end{aligned}$$

With the Feynmann rules given in the previous section, the ξ integral can be performed, and we obtain [27]

$$\begin{aligned}
\frac{x^{\text{eff}}}{x} &= 1 - \frac{1}{F^2} \left(\sum_i^{N_f} \bar{\Delta}(M_{iv}^2|0) - \bar{G}(M_{vv}^2, M_{vv}^2|0) \right), \\
\frac{z_i^{\text{eff}}}{z_i} &= 1 - \frac{1}{F^2} \left(\sum_j^{N_f} \bar{\Delta}(M_{ij}^2|0) - \bar{G}(M_{ii}^2, M_{ii}^2|0) \right). \tag{3.9}
\end{aligned}$$

The corrections are UV divergent, and thus need regularization. Note that each of x and z_i receive different one-loop correction in general. In practice, however, the following three special cases are of our interest:

- (1) Both of the valence quarks and the sea quarks are in the ϵ -regime:
One can to this order set $M_{ij} = 0$ for all i and j , and take the $m_0 \rightarrow \infty$ limit, which leads to

$$\frac{x^{\text{eff}}}{x} = \frac{z_i^{\text{eff}}}{z_i} = 1 - \frac{1}{F^2} \frac{N_f^2 - 1}{N_f} \bar{\Delta}(0|0) \quad (\text{for all } i). \tag{3.10}$$

This correction is equivalent to that of N_f -flavor full theory. In dimensional regularization,

$$\bar{\Delta}(0|0) = -\frac{\beta_1}{L^2} + \mathcal{O}(1/L^4), \quad (3.11)$$

is obtained where β_1 is known as the shape coefficient [4]. It depends only on the shape of the 4-dimensional Euclidean space-time volume.

- (2) The sea quarks are in the p -regime while the valence quarks are still in the ϵ -regime (the m_i 's are heavy but much smaller than the QCD scale, Λ_{QCD}):

One can take the $m_0 \rightarrow \infty$ limit but should keep m_i 's finite, which leads to

$$\begin{aligned} \frac{x^{\text{eff}}}{x} &= 1 - \frac{1}{F^2} \left(\sum_i^{N_f} \bar{\Delta}(M_{iv}^2|0) - \frac{1}{V} \sum_{p \neq 0} \frac{1}{p^4 (\sum_i^{N_f} \frac{1}{p^2 + M_{ii}^2})} \right), \\ \frac{z_i^{\text{eff}}}{z_i} &= 1 - \frac{1}{F^2} \left(\sum_j^{N_f} \bar{\Delta}(M_{ij}^2|0) - \frac{1}{V} \sum_{p \neq 0} \frac{1}{(p^2 + M_{ii}^2)^2 (\sum_j^{N_f} \frac{1}{p^2 + M_{jj}^2})} \right). \end{aligned} \quad (3.12)$$

Note that a double pole contribution appears in x^{eff}/x , as an effect of the partially quenching.

- (3) All the sea quark masses are heavy, $m_i \gg \Lambda_{\text{QCD}}$:

In this case, m_0 cannot be large, but one can take the $m_i \rightarrow \infty$ limit, which leads to

$$\frac{x^{\text{eff}}}{x} = 1 + \frac{1}{F^2} \left(\frac{1}{V} \sum_{p \neq 0} \frac{(m_0^2 + \alpha p^2)/N_c}{p^4} \right), \quad (3.13)$$

which agrees with the quenched result.

To summarize this section, the chiral condensate to one-loop order is given by

$$\Sigma_\nu^{\text{PQ}, 1\text{-loop}}(x, \{z_i\}) = \Sigma_\nu^{\text{PQ}}(x^{\text{eff}}, \{z_i^{\text{eff}}\}) \frac{x^{\text{eff}}}{x}, \quad (3.14)$$

where the analytical functional form of $\Sigma_\nu^{\text{PQ}}(x, \{z_i\})$ is given by Eq.(3.1). When all the quarks are in the ϵ -regime, the one-loop correction is, to this order, constant, and can simply be taken into account in the Lagrangian by shifting Σ according to the above prescription. The chiral condensate in the infinite volume limit, Σ , and all the other low-energy constants, are of course expected to depend on the number of flavors. Matching conditions [28,29,30,31,32] can ensure smooth connections between theories with different number of flavors.

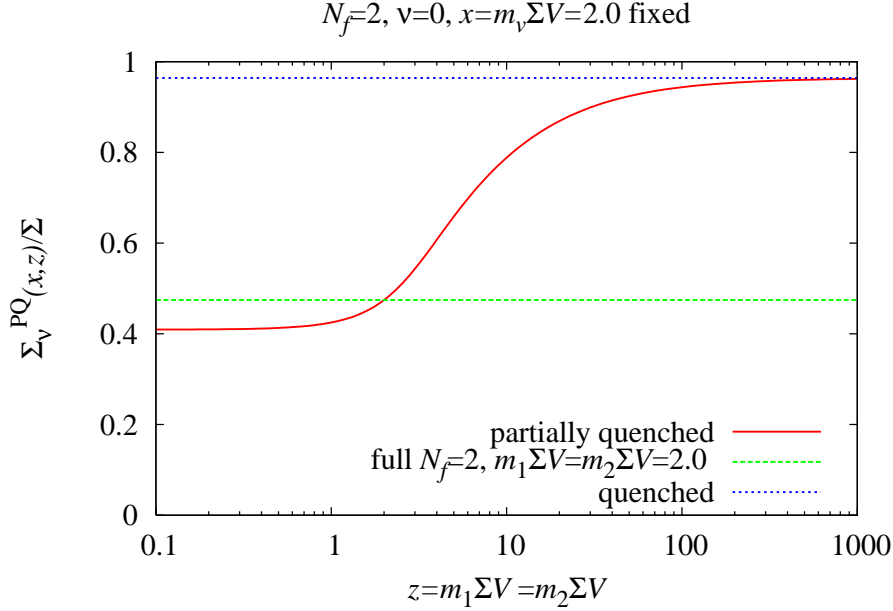


Fig. 1. The sea quark mass dependence of the 2-flavor partially quenched condensate at $\nu = 0$ (solid). The valence quark mass is fixed to $m_v\Sigma V = 2.0$. As expected, we obtain a smooth curve crossing the full 2-flavor theory result (dashed) at $m_1 = m_2 = m_v$, and converging to the quenched limit (dotted).

4 Zero-mode integrals in the partially quenched theory

4.1 $U(N_f + N)$ group integrals in the replica limit

In the ϵ -regime, the integral over the zero-mode, U_0 (which for simplicity of notation will be denoted by U in this subsection) has to be done exactly. This is the central difference between the ϵ -regime and the p -regime, and we are fortunately able to perform the required group integrations exactly. Because these technical aspects are so important for the calculation that follows, we first give a detailed outline of how the integrations have been done, and how we employ the replica formalism in this context. As in Ref. [5] the needed group integrals are conveniently obtained through the identities that follow from the fact that $[\det(\Sigma V \mathcal{M})]^{-\nu} \mathcal{Z}_{N_f+N}^\nu(x, \{z_i\})$ is a function of $\mathcal{M}^\dagger \mathcal{M}$ only. Since the group integrals are known for diagonal sources, this basically solves the problem. We begin, however, with two simpler cases. They do not require any special techniques, and can be evaluated straightforwardly by making use of the graded partition function:

$$\frac{1}{2}\langle(U_{vv} + U_{vv}^\dagger)\rangle = \lim_{y \rightarrow x} \partial_y \ln \mathcal{Z}_{1,1+N_f}^\nu(x|y, \{z_i\}) = \frac{\Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma}, \quad (4.1)$$

$$\begin{aligned} \frac{1}{4}\langle(U_{vv} + U_{vv}^\dagger)^2\rangle &= \frac{1}{\mathcal{Z}_{N_f}^\nu(\{z_i\})} \lim_{y \rightarrow x} \partial_y^2 \mathcal{Z}_{1,1+N_f}^\nu(x|y, \{z_i\}) \\ &= \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} - \frac{\Delta \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma}, \end{aligned} \quad (4.2)$$

where the second term of Eq.(4.2) is defined by

$$\frac{\Delta \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} \equiv \frac{\lim_{y \rightarrow x} \partial_y \partial_x \mathcal{Z}_{1,1+N_f}^\nu(x|y, \{z_i\})}{\mathcal{Z}_{N_f}^\nu(\{z_i\})}. \quad (4.3)$$

If we put $x = z_j$ (for any j), this simply amounts to removing the fermion determinant of quark species j . From the definition (4.2) we then immediately get

$$\begin{aligned} \left[\frac{\partial_x \Sigma_\nu^{\text{PQ}}(z_j, \{z_i\})}{\Sigma} - \frac{\Delta \Sigma_\nu^{\text{PQ}}(z_j, \{z_i\})}{\Sigma} \right]_{x=z_j} &= \\ \partial_{z_j} \frac{\Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} + \left(\frac{\Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right)^2, \end{aligned} \quad (4.4)$$

corresponding to the result in the full theory without partial quenching. Conversely, in the limit $z_i \rightarrow \infty$ (for all i),

$$\begin{aligned} \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} &\xrightarrow{\{z_i\} \rightarrow \infty} \frac{\partial_x \Sigma_\nu^{\text{que}}(x)}{\Sigma}, \\ -\frac{\Delta \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} &\xrightarrow{\{z_i\} \rightarrow \infty} 1 + \frac{\nu^2}{x^2}, \end{aligned} \quad (4.5)$$

we recover the results of the quenched theory [5]. Here, in a hopefully obvious notation, we have denoted the chiral condensates in the full and quenched theories by $\Sigma_\nu^{\text{full}(N_f, j)}$ and Σ_ν^{que} (See Eq.(3.2) and Eq.(3.3)), respectively.

We next consider a purely imaginary source iJ (with J real) on a diagonal (v, v) element of \mathcal{M} ,

$$\Sigma V \mathcal{M} + \mathcal{J} = \text{diag}(\underbrace{z_1, z_2, \dots}_{N_f}, \underbrace{(x + iJ), x, \dots, x}_N). \quad (4.6)$$

We note that

$$\lim_{N \rightarrow 0} \det(\Sigma V \mathcal{M} + \mathcal{J}) = \det(\Sigma V \mathcal{M}) \times (1 + iJ/x), \quad (4.7)$$

and

$$(\Sigma V\mathcal{M} + \mathcal{J})^\dagger(\Sigma V\mathcal{M} + \mathcal{J}) = \text{diag}(\underbrace{z_1^2, z_2^2, \dots}_{N_f}, \underbrace{\lambda^2, x^2, \dots, x^2}_N), \quad (4.8)$$

where $\lambda = \sqrt{x^2 + J^2}$. By the chain rule, J -derivative and λ -derivative are related through

$$\left. \frac{\partial}{\partial J} \right|_{J=0} = \frac{J}{\lambda} \left. \frac{\partial}{\partial \lambda} \right|_{J=0} = 0, \quad \left. \frac{\partial^2}{\partial J^2} \right|_{J=0} = \frac{1}{x} \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=x}. \quad (4.9)$$

From the above equations, we obtain (denote $\mathcal{Z}_{N_f+N}^\nu(x, \{z_i\})$ by \mathcal{Z} for simplicity),

$$\begin{aligned} \frac{1}{\mathcal{Z}} \frac{\partial}{\partial J} \det(\Sigma V\mathcal{M} + \mathcal{J})^{-\nu} \mathcal{Z}|_{J=0} &= \det(\Sigma V\mathcal{M})^{-\nu} \left(-\frac{i\nu}{x} + \frac{1}{\mathcal{Z}} \frac{\partial}{\partial J} \mathcal{Z} \right) = 0, \\ \frac{1}{\mathcal{Z}} \frac{\partial^2}{\partial J^2} \det(\Sigma V\mathcal{M} + \mathcal{J})^{-\nu} \mathcal{Z}|_{J=0} &= \det(\Sigma V\mathcal{M})^{-\nu} \left(\frac{\nu(\nu-1)}{x^2} + \left. \frac{1}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial J^2} \right|_{J=0} \right), \\ \frac{1}{\mathcal{Z}} \frac{1}{x} \frac{\partial}{\partial \lambda} \det(\Sigma V\mathcal{M} + \mathcal{J})^{-\nu} \mathcal{Z}|_{\lambda=x} &= \det(\Sigma V\mathcal{M})^{-\nu} \left(-\frac{\nu}{x^2} + \frac{\Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{x\Sigma} \right). \end{aligned} \quad (4.10)$$

This gives us the useful identities

$$\frac{1}{2} \langle (U_{vv} - U_{vv}^\dagger) \rangle = i \left. \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial J} \right|_{J=0} = -\frac{\nu}{x} \quad (4.11)$$

$$\frac{1}{4} \langle (U_{vv} - U_{vv}^\dagger)^2 \rangle = - \left. \frac{1}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial J^2} \right|_{J=0} = -\frac{\Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{x\Sigma} + \frac{\nu^2}{x^2}. \quad (4.12)$$

In order to calculate the meson correlators in PQChPT, we also need matrix elements which have different valence flavor indices. For example, for the disconnected correlators, we need

$$\begin{aligned} &\frac{1}{4} \langle (U_{v_1 v_1} + U_{v_1 v_1}^\dagger)(U_{v_2 v_2} + U_{v_2 v_2}^\dagger) \rangle \\ &= \frac{1}{\mathcal{Z}_{N_f}^\nu(\{z_i\})} \lim_{y_1 \rightarrow x_1, y_2 \rightarrow x_2} \partial_{y_1} \partial_{y_2} \mathcal{Z}_{2,2+N_f}^\nu(x_1, x_2 | y_1, y_2, \{z_i\}) \\ &\equiv D_\nu^{\text{PQ}}(x_1, x_2, \{z_i\}). \end{aligned} \quad (4.13)$$

To derive the analytical expressions for these and others closely related, we first consider two purely imaginary sources $\mathcal{J}_{v_1 v_1} = iJ_1$ and $\mathcal{J}_{v_2 v_2} = iJ_2$ along

the diagonal:

$$\Sigma V\mathcal{M} + \mathcal{J} = \text{diag}(\underbrace{z_1, z_2, \dots}_{N_f}, \underbrace{(x_1 + iJ_1), (x_2 + iJ_2), x, \dots, x}_N), \quad (4.14)$$

The replica limit of this determinant is

$$\lim_{N \rightarrow 0} \det(\Sigma V\mathcal{M} + \mathcal{J}) = \det(\Sigma V\mathcal{M}) \times (1 + iJ_1/x_1)(1 + iJ_2/x_2), \quad (4.15)$$

and $(\Sigma V\mathcal{M} + \mathcal{J})^\dagger(\Sigma V\mathcal{M} + \mathcal{J})$ takes the form

$$(\Sigma V\mathcal{M} + \mathcal{J})^\dagger(\Sigma V\mathcal{M} + \mathcal{J}) \rightarrow \text{diag}(\underbrace{z_1^2, z_2^2, \dots}_{N_f}, \underbrace{\lambda_1^2, \lambda_2^2, x^2, \dots, x^2}_N), \quad (4.16)$$

where $\lambda_i = \sqrt{x_i^2 + J_i^2}$ ($i = 1, 2$). Again, J -derivative and λ -derivative are related through the chain rule,

$$\left. \frac{\partial \lambda_i}{\partial J_i} \right|_{J_i=0} = \left. \frac{J_i}{\lambda_i} \right|_{J_i=0} = 0. \quad (4.17)$$

From these equations we get

$$\begin{aligned} & \frac{1}{\mathcal{Z}} \frac{\partial^2}{\partial J_1 \partial J_2} \det(\Sigma V\mathcal{M} + \mathcal{J})^{-\nu} \mathcal{Z} \Big|_{\mathcal{J}=0} \\ &= \det(\Sigma V\mathcal{M})^{-\nu} \left(\frac{-\nu^2}{x_1 x_2} + \frac{1}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial J_1 \partial J_2} \Big|_{\mathcal{J}=0} \right) = 0, \\ & \frac{1}{4} \langle (U_{v_1 v_1} - U_{v_1 v_1}^\dagger)(U_{v_2 v_2} - U_{v_2 v_2}^\dagger) \rangle = \frac{\nu^2}{x_1 x_2}. \end{aligned} \quad (4.18)$$

Next let us consider a real off-diagonal source, $\mathcal{J}_{v_1 v_2} = J$, for which the determinant is unchanged in the replica limit:

$$\lim_{N \rightarrow 0} \det(\Sigma V\mathcal{M} + \mathcal{J}) = \det(\Sigma V\mathcal{M}). \quad (4.19)$$

Now $(\Sigma V\mathcal{M} + \mathcal{J})^\dagger(\Sigma V\mathcal{M} + \mathcal{J})$ can be diagonalized as

$$(\Sigma V\mathcal{M} + \mathcal{J})^\dagger(\Sigma V\mathcal{M} + \mathcal{J}) \rightarrow \text{diag}(\underbrace{z_1^2, z_2^2, \dots}_{N_f}, \underbrace{\lambda_+^2, \lambda_-^2, x^2, \dots, x^2}_N), \quad (4.20)$$

where

$$\lambda_{\pm} = \sqrt{\frac{(J^2 + x_1^2 + x_2^2) \pm \sqrt{J^4 + 2J^2(x_1^2 + x_2^2) + (x_1^2 - x_2^2)^2}}{2}}. \quad (4.21)$$

If we assume that $x_1 \neq x_2$ (the special case $x_1 \rightarrow x_2$ [5] can be taken as a limiting case afterwards, see below), we obtain the following relation between J -derivatives and λ -derivatives:

$$\left. \frac{\partial^2}{\partial J^2} \right|_{J=0} = \frac{1}{x_1^2 - x_2^2} \left(x_1 \left. \frac{\partial}{\partial \lambda_+} \right|_{\lambda_+ = x_1} - x_2 \left. \frac{\partial}{\partial \lambda_-} \right|_{\lambda_- = x_2} \right), \quad (4.22)$$

We now use

$$\begin{aligned} \frac{1}{\mathcal{Z}} \frac{1}{x_1^2 - x_2^2} \left(x_1 \frac{\partial}{\partial \lambda_+} - x_2 \frac{\partial}{\partial \lambda_-} \right) \mathcal{Z} \Big|_{\lambda_+ = x_1, \lambda_- = x_2} = \\ \frac{1}{x_1^2 - x_2^2} \left(x_1 \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - x_2 \frac{\Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} \right). \end{aligned} \quad (4.23)$$

Note that a purely imaginary source $\mathcal{J}_{v_1 v_2} = iJ$ gives the same results. Thus, one obtains

$$\frac{1}{4} \langle (U_{v_1 v_2} \pm U_{v_2 v_1}^\dagger)^2 \rangle = \frac{\pm}{x_1^2 - x_2^2} \left(x_1 \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - x_2 \frac{\Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} \right). \quad (4.24)$$

As a check, if we take the mass-degenerate limit $x_2 \rightarrow x_1$ we recover

$$\frac{1}{4} \langle (U_{v_1 v_2} \pm U_{v_2 v_1}^\dagger)^2 \rangle = \pm \frac{1}{2} \left(\frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{x_1 \Sigma} + \frac{\partial_{x_1} \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} \right), \quad (4.25)$$

which is obtained from the formula in the degenerate case in Eq.(4.22),

$$\left. \frac{\partial^2}{\partial J^2} \right|_{J=0} = \frac{1}{4x_1} \left(\frac{\partial}{\partial \lambda_+} + \frac{\partial}{\partial \lambda_-} \right) + \frac{1}{4} \left(\frac{\partial}{\partial \lambda_+} - \frac{\partial}{\partial \lambda_-} \right)^2 \Big|_{\lambda_\pm = x_1}, \quad (4.26)$$

where $\lambda_\pm = (\sqrt{J^2 + 4x_1^2} \pm J)/2$.

We finally put two real sources on the off-diagonal elements, $\mathcal{J}_{v_1 v_2} = J_1$ and $\mathcal{J}_{v_2 v_1} = J_2$. In the replica limit the determinant becomes

$$\lim_{N \rightarrow 0} \det(\Sigma V \mathcal{M} + \mathcal{J}) = \det(\Sigma V \mathcal{M}) (1 - J_1 J_2 / x_1 x_2), \quad (4.27)$$

and $(\Sigma V \mathcal{M} + \mathcal{J})^\dagger (\Sigma V \mathcal{M} + \mathcal{J})$ diagonalizes as

$$(\Sigma V \mathcal{M} + \mathcal{J})^\dagger (\Sigma V \mathcal{M} + \mathcal{J}) \rightarrow \text{diag}(\underbrace{z_1^2, z_2^2, \dots}_{N_f}, \underbrace{\lambda_+^2, \lambda_-^2, x^2, \dots}_N, x^2), \quad (4.28)$$

where

$$\lambda_{\pm} = \sqrt{\frac{(J_1^2 + J_2^2 + x_1^2 + x_2^2) \pm \sqrt{(J_1^2 + J_2^2 + x_1^2 + x_2^2)^2 - 4(J_1 J_2 - x_1 x_2)^2}}{2}}. \quad (4.29)$$

The relation between J -derivative and λ -derivative can be worked out as above. Assuming again that $x_1 \neq x_2$ (the degenerate case can also here be recovered by taking the limit $x_1 \rightarrow x_2$ afterwards), we get

$$\left. \frac{\partial^2}{\partial J_1 \partial J_2} \right|_{J_i=0} = \frac{1}{x_1^2 - x_2^2} \left(x_2 \left. \frac{\partial}{\partial \lambda_+} \right|_{\lambda_+=x_1} - x_1 \left. \frac{\partial}{\partial \lambda_-} \right|_{\lambda_-=x_2} \right). \quad (4.30)$$

In the same way as above, we thus find that the two equations

$$\begin{aligned} \frac{1}{\mathcal{Z}} \frac{\partial^2}{\partial J_1 \partial J_2} \det(\Sigma V \mathcal{M} + \mathcal{J})^{-\nu} \mathcal{Z} \Big|_{\mathcal{J}=0} &= \\ \det(\Sigma V \mathcal{M})^{-\nu} \left(\frac{\nu}{x_1 x_2} + \frac{1}{\mathcal{Z}} \left. \frac{\partial^2 \mathcal{Z}}{\partial J_1 \partial J_2} \right|_{\mathcal{J}=0} \right), \\ \frac{1}{\mathcal{Z}} \frac{1}{x_1^2 - x_2^2} \left(x_2 \left. \frac{\partial}{\partial \lambda_+} \right|_{\lambda_+=x_1} - x_1 \left. \frac{\partial}{\partial \lambda_-} \right|_{\lambda_-=x_2} \right) \det(\Sigma V \mathcal{M} + \mathcal{J})^{-\nu} \mathcal{Z} \Big|_{\lambda_i=x_i} &= \det(\Sigma V \mathcal{M})^{-\nu} \\ \times \left(\frac{\nu}{x_1 x_2} + \frac{1}{x_1^2 - x_2^2} \left(x_2 \frac{\Sigma_{\nu}^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - x_1 \frac{\Sigma_{\nu}^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} \right) \right), \end{aligned} \quad (4.31)$$

lead to

$$\begin{aligned} \frac{1}{4} \langle (U_{v_1 v_2} \pm U_{v_2 v_1}^{\dagger})(U_{v_2 v_1} \pm U_{v_1 v_2}^{\dagger}) \rangle &= \\ \frac{1}{x_1^2 - x_2^2} \left(x_2 \frac{\Sigma_{\nu}^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - x_1 \frac{\Sigma_{\nu}^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} \right), \end{aligned} \quad (4.32)$$

where we used that purely imaginary sources $\mathcal{J}_{v_1 v_2} = iJ_1$ $\mathcal{J}_{v_2 v_1} = -iJ_2$ give the same result. We summarize all pertinent formulas in appendix A.

4.2 The Unitarity Formula

From the above formulae and the requirement of unitarity one finds in the replica limit,

$$\begin{aligned}
1 &= \left\langle \lim_{N \rightarrow 0} \sum_i^{N+N_f} U_{v_1 i}^\dagger U_{i v_1} \right\rangle \\
&= \left\langle \lim_{N \rightarrow 0} (U_{v_1 v_1}^\dagger U_{v_1 v_1} + \underbrace{U_{v_1 v_2}^\dagger U_{v_2 v_1} + \cdots + U_{v_1 v_N}^\dagger U_{v_N v_1}}_{N-1}) + \sum_i^{N_f} U_{v_1 i}^\dagger U_{i v_1} \right\rangle \\
&= \left\langle U_{v_1 v_1}^\dagger U_{v_1 v_1} - U_{v_1 v_2}^\dagger U_{v_2 v_1} + \sum_i^{N_f} U_{v_1 i}^\dagger U_{i v_1} \right\rangle, \tag{4.33}
\end{aligned}$$

where we have used that the N replicated quarks are all degenerate: $x_i = m_{v_i} \Sigma V = x$. One then obtains a non-trivial identity,

$$\begin{aligned}
&\frac{\Delta \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} + 1 + \frac{\nu^2}{x^2} = \\
&\sum_{j=1}^{N_f} \frac{2}{x^2 - z_j^2} \left[\frac{x \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right]. \tag{4.34}
\end{aligned}$$

It is not difficult to show that the left hand side of Eq.(4.34) actually vanishes in the quenched limit, *cf.* Eq.(4.5).

It is also interesting to consider the limit of the full N_f -flavor theory of Eq.(4.34), in which all the valence and sea quarks are degenerate: $x = z_1 = z_2 = \cdots = z$. In that degenerate limit we have

$$\begin{aligned}
\frac{\Sigma_\nu^{N_f}(z)}{\Sigma} &\equiv \frac{1}{N_f} \sum_i \frac{\Sigma_\nu^{\text{full}(N_f, i)}(z_1, z_2, \cdots)}{\Sigma} \Big|_{z_1=z_2=\cdots=z}, \tag{4.35} \\
\left(\frac{\Sigma_\nu^{N_f}(z)}{\Sigma} \right)' &= \frac{\partial_{z_1} \Sigma_\nu^{\text{full}(N_f, 1)}(z_1, z_2, \cdots)}{\Sigma} \Big|_{z_1=z_2=\cdots=z} \\
&\quad + (N_f - 1) \frac{\partial_{z_2} \Sigma_\nu^{\text{full}(N_f, 1)}(z_1, z_2, \cdots)}{\Sigma} \Big|_{z_1=z_2=\cdots=z}, \tag{4.36}
\end{aligned}$$

where we have used the fact that the $\partial_{z_j} \Sigma_\nu^{\text{full}(i)}$'s are independent of i or j in the degenerate case. We thus have

$$\left. \frac{\partial_{z_2} \Sigma_\nu^{\text{full}(N_f, 1)}(z_1, z_2, \dots)}{\Sigma} \right|_{z_1=z_2=\dots=z} = \frac{1}{N_f - 1} \left(\left(\frac{\Sigma_\nu^{N_f}(z)}{\Sigma} \right)' - \left. \frac{\partial_{z_1} \Sigma_\nu^{\text{full}(N_f, 1)}(z_1, z_2, \dots)}{\Sigma} \right|_{z_1=z_2=\dots=z} \right). \quad (4.37)$$

Similarly, one obtains

$$\left. \frac{\partial_{z_1} \Sigma_\nu^{\text{full}(N_f, 1)}(z_1, z_2, \dots)}{\Sigma} \right|_{z_1=z_2=\dots=z} = \frac{N_f - 1}{N_f} \left. \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z, z, \dots\})}{\Sigma} \right|_{x=z} + \frac{1}{N_f} \left(\frac{\Sigma_\nu^{N_f}(z)}{\Sigma} \right)'. \quad (4.38)$$

The unitarity equation for the degenerate case ($x = z_1 = z_2 = \dots = z$), Eq.(4.34) then finally becomes

$$\begin{aligned} & - \left(\frac{\Sigma_\nu^{N_f}(z)}{\Sigma} \right)^2 - \frac{1}{N_f} \left(\frac{\Sigma_\nu^{N_f}(z)}{\Sigma} \right)' - N_f \frac{\Sigma_\nu^{N_f}(z)}{z \Sigma} + 1 + \frac{\nu^2}{z^2} \\ & = \frac{(N_f^2 - 1)}{N_f} \left. \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z, z, \dots\})}{\Sigma} \right|_{x=z} \\ & = (N_f + 1) \left(\left. \frac{\partial_{z_1} \Sigma_\nu^{\text{full}(N_f, 1)}(\{z_1, z, z, \dots\})}{\Sigma} \right|_{z_1=z} - \frac{\Sigma'_\nu(z)}{N_f \Sigma} \right) \end{aligned} \quad (4.39)$$

a relation which is useful when simplifying the expressions for the meson correlators in Sec.5. We finally note that Eq.(4.39) is consistent with an analogous formula obtained from Schwinger-Dyson equations [5] in the full theory.

5 Meson correlators

In this section we present the detailed analytical predictions of partially quenched scalar and pseudoscalar correlation functions in the ϵ -regime of QCD. Our calculation is done to the lowest non-trivial order in the ϵ -expansion, and by taking limits we can check that we recover both the fully quenched and $N_f = 2$ results at sectors of fixed topological index ν as reported in ref. [5].

5.1 Partially Quenched Correlators with Two Valence Quarks

Of most immediate interest are the correlation functions with two light valence quarks, representing the u and d quarks in QCD. The physical case of low-energy QCD with two very light quarks (the u and the d quarks) and one heavier quark (the s quark) is an example where one can be in a mixture of the ϵ -regime (with respect to the two light quarks) and the p -regime (with respect to the s quark). Even if the two light quarks correspond, for the given lattice volume, to the ϵ -regime, it may be convenient to recycle the lattice configurations by considering light valence quarks that are *also* in the ϵ -regime, but just at different mass values. This requires a comparison with the results of PQChPT in the ϵ -regime that we shall present here.

Furthermore, with the resulting formulae as building blocks one can calculate various other different types of meson correlators. They can correspond not just to different valence and sea quark masses but also to an arbitrary number of valence and sea quark flavors. For example, the $SU(3)$ -singlet scalar correlator is obtained by

$$\langle S^0(x) S^0(0) \rangle = \sum_i^3 \sum_j^3 \langle \bar{q}_{v_i} q_{v_i}(x) \bar{q}_{v_j} q_{v_j}(0) \rangle_d + \sum_i^3 \langle \bar{q}_{v_i} q_{v_i}(x) \bar{q}_{v_i} q_{v_i}(0) \rangle_c, \quad (5.1)$$

where the expressions in the r.h.s. are defined below. Note that the contributions inside the sums can differ due to the non-degeneracy of the valence quark masses.

The connected scalar and pseudo-scalar correlators are defined by

$$\begin{aligned} & \langle \bar{q}_{v_1} q_{v_2}(x) \bar{q}_{v_2} q_{v_1}(0) \rangle_c \\ & \equiv \frac{\Sigma^2}{8} \left\langle \left(U(x)_{v_1 v_2} + U^\dagger(x)_{v_2 v_1} + U(x)_{v_2 v_1} + U^\dagger(x)_{v_1 v_2} \right) \right. \\ & \quad \times \left. \left(U(0)_{v_1 v_2} + U^\dagger(0)_{v_2 v_1} + U(0)_{v_2 v_1} + U^\dagger(0)_{v_1 v_2} \right) \right\rangle, \end{aligned} \quad (5.2)$$

$$\begin{aligned} & \langle \bar{q}_{v_1} \gamma_5 q_{v_2}(x) \bar{q}_{v_2} \gamma_5 q_{v_1}(0) \rangle_c \\ & \equiv \frac{\Sigma^2}{8} \left\langle \left(U(x)_{v_1 v_2} - U^\dagger(x)_{v_2 v_1} + U(x)_{v_2 v_1} - U^\dagger(x)_{v_1 v_2} \right) \right. \\ & \quad \times \left. \left(U(0)_{v_1 v_2} - U^\dagger(0)_{v_2 v_1} + U(0)_{v_2 v_1} - U^\dagger(0)_{v_1 v_2} \right) \right\rangle, \end{aligned} \quad (5.3)$$

where v_i denotes the valence flavor index with the quark mass $m_{v_i} = x_i/\Sigma V$. Note that Eq.(5.2) and Eq.(5.3) have the same structure as the iso-triplet correlators in the 2-flavor theory of which masses are m_{v_1} and m_{v_2} .

The disconnected correlators are similarly defined

$$\begin{aligned}
& \langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_2} q_{v_2}(0) \rangle_d \\
& \equiv \frac{\Sigma^2}{4} \left\langle \left(U(x)_{v_1 v_1} + U^\dagger(x)_{v_1 v_1} \right) \left(U(0)_{v_2 v_2} + U^\dagger(0)_{v_2 v_2} \right) \right\rangle,
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
& \langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_2} \gamma_5 q_{v_2}(0) \rangle_d \\
& \equiv \frac{\Sigma^2}{4} \left\langle \left(U(x)_{v_1 v_1} - U^\dagger(x)_{v_1 v_1} \right) \left(U(0)_{v_2 v_2} - U^\dagger(0)_{v_2 v_2} \right) \right\rangle.
\end{aligned} \tag{5.5}$$

Let us begin with the disconnected scalar correlators. To $O(\epsilon^2)$, we find

$$\begin{aligned}
& \langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_2} q_{v_2}(0) \rangle_d \\
& = \frac{\Sigma^2}{4} \left\langle (U_{v_1 v_1} + U_{v_1 v_1}^\dagger)(U_{v_2 v_2} + U_{v_2 v_2}^\dagger) \right\rangle^{1\text{-loop}} \times \left(\frac{x_1^{\text{eff}}}{x_1} \right) \left(\frac{x_2^{\text{eff}}}{x_2} \right) \\
& \quad - \frac{\Sigma^2}{2F^2} \langle U_{v_1 v_2} U_{v_2 v_1} + U_{v_2 v_1}^\dagger U_{v_1 v_2}^\dagger \rangle \bar{P}_{(12)(21)}(x) \\
& \quad - \frac{\Sigma^2}{2F^2} \langle (U_{v_1 v_1} - U_{v_1 v_1}^\dagger)(U_{v_2 v_2} - U_{v_2 v_2}^\dagger) \rangle \bar{P}_{(11)(22)}(x) \\
& = \frac{\Sigma^2}{4} \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left\langle (U_{v_1 v_1} + U_{v_1 v_1}^\dagger)(U_{v_2 v_2} + U_{v_2 v_2}^\dagger) \right\rangle^{1\text{-loop}} \\
& \quad - \frac{\Sigma^2}{2F^2} \langle U_{v_1 v_2} U_{v_2 v_1} + U_{v_2 v_1}^\dagger U_{v_1 v_2}^\dagger \rangle \bar{\Delta}(0|x) \\
& \quad - \frac{\Sigma^2}{2F^2} \langle (U_{v_1 v_1} - U_{v_1 v_1}^\dagger)(U_{v_2 v_2} - U_{v_2 v_2}^\dagger) \rangle \bar{G}(0, 0|x),
\end{aligned} \tag{5.6}$$

We have consistently set $m_{v_1} = m_{v_2} = 0$ in the NLO contributions, and

$$\frac{x_1^{\text{eff}}}{x_1} = \frac{x_2^{\text{eff}}}{x_2} = 1 - \frac{1}{F^2} \left(\sum_i^{N_f} \bar{\Delta}(M_{iv}^2|0) - \bar{G}(0, 0|0) \right), \tag{5.7}$$

which of course needs regularization. Note $\langle \dots \rangle^{1\text{-loop}}$ indicates the shift $x_i \rightarrow x_i^{\text{eff}}$ and $z_i \rightarrow z_i^{\text{eff}}$ in the arguments of the Bessel functions.

In the same way we obtain the disconnected pseudo-scalar correlation function

$$\begin{aligned}
& \langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_2} \gamma_5 q_{v_2}(0) \rangle_d \\
&= \frac{\Sigma^2}{4} \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left\langle (U_{v_1 v_1} - U_{v_1 v_1}^\dagger)(U_{v_2 v_2} - U_{v_2 v_2}^\dagger) \right\rangle^{1\text{-loop}} \\
&\quad - \frac{\Sigma^2}{2F^2} \langle U_{v_1 v_2} U_{v_2 v_1} + U_{v_2 v_1}^\dagger U_{v_1 v_2}^\dagger \rangle \bar{\Delta}(0|x) \\
&\quad - \frac{\Sigma^2}{2F^2} \langle (U_{v_1 v_1} + U_{v_1 v_1}^\dagger)(U_{v_2 v_2} + U_{v_2 v_2}^\dagger) \rangle \bar{G}(0, 0|x).
\end{aligned} \tag{5.8}$$

For the connected correlators, we get

$$\begin{aligned}
& \langle \bar{q}_{v_1} q_{v_2}(x) \bar{q}_{v_2} q_{v_1}(0) \rangle_c \\
&= \frac{\Sigma^2}{4} \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left\langle (U_{v_1 v_2} + U_{v_2 v_1}^\dagger)^2 + (U_{v_1 v_2} + U_{v_2 v_1}^\dagger)(U_{v_2 v_1} + U_{v_1 v_2}^\dagger) \right\rangle^{1\text{-loop}} \\
&\quad + \frac{\Sigma^2}{2F^2} \sum_i' \left[\langle U_{v_1 i} U_{i v_1}^\dagger \rangle \bar{P}_{(2i)(i2)}(x) + \langle U_{v_2 i} U_{i v_2}^\dagger \rangle \bar{P}_{(1i)(i1)}(x) \right] \\
&\quad - \frac{\Sigma^2}{4F^2} \left[\langle U_{v_1 v_2}^2 + (U_{v_2 v_1}^\dagger)^2 \rangle \bar{P}_{(22)(22)}(x) + \langle U_{v_2 v_1}^2 + (U_{v_1 v_2}^\dagger)^2 \rangle \bar{P}_{(11)(11)}(x) \right] \\
&\quad - \frac{\Sigma^2}{4F^2} \langle U_{v_1 v_1} U_{v_2 v_2} + U_{v_1 v_1}^\dagger U_{v_2 v_2}^\dagger \rangle \left[\bar{P}_{(12)(21)}(x) + \bar{P}_{(21)(12)}(x) \right] \\
&\quad - \frac{\Sigma^2}{4F^2} \langle (U_{v_1 v_2} - U_{v_2 v_1}^\dagger)(U_{v_2 v_1} - U_{v_1 v_2}^\dagger) \rangle \bar{P}_{(22)(11)}(x) \\
&\quad - \frac{\Sigma^2}{4F^2} \langle (U_{v_2 v_1} - U_{v_1 v_2}^\dagger)(U_{v_1 v_2} - U_{v_2 v_1}^\dagger) \rangle \bar{P}_{(11)(22)}(x),
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
& \langle \bar{q}_{v_1} \gamma_5 q_{v_2}(x) \bar{q}_{v_2} \gamma_5 q_{v_1}(0) \rangle_c \\
&= \frac{\Sigma^2}{4} \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left\langle (U_{v_1 v_2} - U_{v_2 v_1}^\dagger)^2 + (U_{v_1 v_2} - U_{v_2 v_1}^\dagger)(U_{v_2 v_1} - U_{v_1 v_2}^\dagger) \right\rangle^{1\text{-loop}} \\
&\quad - \frac{\Sigma^2}{2F^2} \sum_i' \left[\langle U_{v_1 i} U_{i v_1}^\dagger \rangle \bar{P}_{(2i)(i2)}(x) + \langle U_{v_2 i} U_{i v_2}^\dagger \rangle \bar{P}_{(1i)(i1)}(x) \right] \\
&\quad - \frac{\Sigma^2}{4F^2} \left[\langle U_{v_1 v_2}^2 + (U_{v_2 v_1}^\dagger)^2 \rangle \bar{P}_{(22)(22)}(x) + \langle U_{v_2 v_1}^2 + (U_{v_1 v_2}^\dagger)^2 \rangle \bar{P}_{(11)(11)}(x) \right] \\
&\quad - \frac{\Sigma^2}{4F^2} \langle U_{v_1 v_1} U_{v_2 v_2} + U_{v_1 v_1}^\dagger U_{v_2 v_2}^\dagger \rangle \left[\bar{P}_{(12)(21)}(x) + \bar{P}_{(21)(12)}(x) \right] \\
&\quad - \frac{\Sigma^2}{4F^2} \langle (U_{v_1 v_2} + U_{v_2 v_1}^\dagger)(U_{v_2 v_1} + U_{v_1 v_2}^\dagger) \rangle \bar{P}_{(22)(11)}(x) \\
&\quad - \frac{\Sigma^2}{4F^2} \langle (U_{v_2 v_1} + U_{v_1 v_2}^\dagger)(U_{v_1 v_2} + U_{v_2 v_1}^\dagger) \rangle \bar{P}_{(11)(22)}(x),
\end{aligned} \tag{5.10}$$

where the summation over the flavors, denoted by \sum_i' , has to be taken carefully.

For example, in the sum

$$\begin{aligned}
& \sum_i^I \langle U_{v_1 i} U_{iv_1}^\dagger \rangle \bar{P}_{(2i)(i2)}(x) \\
& \equiv \lim_{N_{v_1} \rightarrow 0} \sum_i^{N_{v_1}} \langle U_{v_1 i} U_{iv_1}^\dagger \rangle \bar{P}_{(2i)(i2)}(x) + \lim_{N_{v_2} \rightarrow 0} \sum_i^{N_{v_2}} \langle U_{v_1 i} U_{iv_1}^\dagger \rangle \bar{P}_{(2i)(i2)}(x) \\
& \quad + \sum_i^{N_f} \langle U_{v_1 i} U_{iv_1}^\dagger \rangle \bar{P}_{(2i)(i2)}(x) \\
& = \langle U_{v_1 v_1} U_{v_1 v_1}^\dagger \rangle \bar{P}_{(21)(12)}(x) - \langle U_{v_1 v_1'} U_{v_1' v_1}^\dagger \rangle \bar{P}_{(21)(12)}(x) \\
& \quad + \langle U_{v_1 v_2} U_{v_2 v_1}^\dagger \rangle \bar{P}_{(22)(22)}(x) - \langle U_{v_1 v_2} U_{v_2 v_1}^\dagger \rangle \bar{P}_{(22')(2'2)}(x) \\
& \quad + \sum_i^{N_f} \langle U_{v_1 i} U_{iv_1}^\dagger \rangle \bar{P}_{(2i)(i2)}(x), \tag{5.11}
\end{aligned}$$

the index with a prime i' is treated as a different flavor from i -th quark but with the same value of the quark mass. With this, we find the correlators

$$\begin{aligned}
& \langle \bar{q}_{v_1} q_{v_2}(x) \bar{q}_{v_2} q_{v_1}(0) \rangle_c \\
& = \frac{\Sigma^2}{4} \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left\langle (U_{v_1 v_2} + U_{v_2 v_1}^\dagger)^2 + (U_{v_1 v_2} + U_{v_2 v_1}^\dagger)(U_{v_2 v_1} + U_{v_1 v_2}^\dagger) \right\rangle^{1\text{-loop}} \\
& \quad + \frac{\Sigma^2}{2F^2} \langle U_{v_1 v_1} U_{v_1 v_1}^\dagger - U_{v_1 v_1'} U_{v_1' v_1}^\dagger + U_{v_2 v_2} U_{v_2 v_2}^\dagger - U_{v_2 v_2'} U_{v_2' v_2}^\dagger \\
& \quad \quad - U_{v_1 v_1} U_{v_2 v_2} - U_{v_1 v_1}^\dagger U_{v_2 v_2}^\dagger \rangle \bar{\Delta}(0|x) \\
& \quad - \frac{\Sigma^2}{2F^2} \langle U_{v_1 v_2} U_{v_2 v_1}^\dagger + U_{v_2 v_1} U_{v_1 v_2}^\dagger - (U_{v_1 v_2} - U_{v_2 v_1}^\dagger)(U_{v_2 v_1} - U_{v_1 v_2}^\dagger) \rangle \bar{G}(0, 0|x) \\
& \quad + \frac{\Sigma^2}{2F^2} \sum_i^{N_f} \langle U_{v_1 i} U_{iv_1}^\dagger + U_{v_2 i} U_{iv_2}^\dagger \rangle \bar{\Delta}(M_{iv}^2|x), \tag{5.12}
\end{aligned}$$

$$\begin{aligned}
& \langle \bar{q}_{v_1} \gamma_5 q_{v_2}(x) \bar{q}_{v_2} \gamma_5 q_{v_1}(0) \rangle_c \\
& = \frac{\Sigma^2}{4} \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left\langle (U_{v_1 v_2} - U_{v_2 v_1}^\dagger)^2 + (U_{v_1 v_2} - U_{v_2 v_1}^\dagger)(U_{v_2 v_1} - U_{v_1 v_2}^\dagger) \right\rangle^{1\text{-loop}} \\
& \quad - \frac{\Sigma^2}{2F^2} \langle U_{v_1 v_1} U_{v_1 v_1}^\dagger - U_{v_1 v_1'} U_{v_1' v_1}^\dagger + U_{v_2 v_2} U_{v_2 v_2}^\dagger - U_{v_2 v_2'} U_{v_2' v_2}^\dagger \\
& \quad \quad + U_{v_1 v_1} U_{v_2 v_2} + U_{v_1 v_1}^\dagger U_{v_2 v_2}^\dagger \rangle \bar{\Delta}(0|x) \\
& \quad + \frac{\Sigma^2}{2F^2} \langle U_{v_1 v_2} U_{v_2 v_1}^\dagger + U_{v_2 v_1} U_{v_1 v_2}^\dagger + (U_{v_1 v_2} + U_{v_2 v_1}^\dagger)(U_{v_2 v_1} + U_{v_1 v_2}^\dagger) \rangle \bar{G}(0, 0|x) \\
& \quad - \frac{\Sigma^2}{2F^2} \sum_i^{N_f} \langle U_{v_1 i} U_{iv_1}^\dagger + U_{v_2 i} U_{iv_2}^\dagger \rangle \bar{\Delta}(M_{iv}^2|x), \tag{5.13}
\end{aligned}$$

where we have explicitly kept the sea quark mass, in order to possibly apply these equations outside of the ϵ -regime.

With aid of the formulae collected in Appendix A and use of the defining equation (4.13) for D_ν^{PQ} , one obtains

$$\begin{aligned}
\langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_2} q_{v_2}(0) \rangle_d &= \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 D_\nu^{\text{PQ}}(x_1^{\text{eff}}, x_2^{\text{eff}}, \{z_i^{\text{eff}}\}) \\
&\quad - \frac{\Sigma^2}{F^2} \frac{2}{x_1^2 - x_2^2} \left(\frac{x_2 \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{x_1 \Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} \right) \bar{\Delta}(0|x) \\
&\quad + \frac{\Sigma^2}{F^2} \frac{2\nu^2}{x_1 x_2} \bar{G}(0, 0|x), \tag{5.14}
\end{aligned}$$

$$\begin{aligned}
\langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_2} \gamma_5 q_{v_2}(0) \rangle_d &= \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \frac{\nu^2}{x_1^{\text{eff}} x_2^{\text{eff}}} \\
&\quad - \frac{\Sigma^2}{F^2} \frac{2}{x_1^2 - x_2^2} \left(\frac{x_2 \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{x_1 \Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} \right) \bar{\Delta}(0|x) \\
&\quad + \frac{2\Sigma^2}{F^2} D_\nu^{\text{PQ}}(x_1, x_2, \{z_i\}) \bar{G}(0, 0|x), \tag{5.15}
\end{aligned}$$

$$\begin{aligned}
\langle \bar{q}_{v_1} q_{v_2}(x) \bar{q}_{v_2} q_{v_1}(0) \rangle_c &= \\
&\quad \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left[\frac{1}{x_1^{\text{eff}} - x_2^{\text{eff}}} \left(\frac{\Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\})}{\Sigma} - \frac{\Sigma_\nu^{\text{PQ}}(x_2^{\text{eff}}, \{z_i^{\text{eff}}\})}{\Sigma} \right) \right] \\
&\quad - \frac{\Sigma^2}{2F^2} \left(\frac{\Delta \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} + \frac{\Delta \Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} + \frac{\nu^2}{x_1^2} + \frac{\nu^2}{x_2^2} \right. \\
&\quad \quad \left. + 2D_\nu^{\text{PQ}}(x_1, x_2, \{z_i\}) + \frac{2\nu^2}{x_1 x_2} \right) \bar{\Delta}(0|x) \\
&\quad - \frac{\Sigma^2}{2F^2} \left[\frac{4}{x_1 + x_2} \left(\frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} + \frac{\Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} \right) \right] \bar{G}(0, 0|x) \\
&\quad + \frac{\Sigma^2}{2F^2} \sum_j^{N_f} \left[\frac{2}{x_1^2 - z_j^2} \left(\frac{x_1 \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right) \right. \\
&\quad \quad \left. + \frac{2}{x_2^2 - z_j^2} \left(\frac{x_2 \Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right) \right] \bar{\Delta}(M_{jv}^2|x), \tag{5.16}
\end{aligned}$$

$$\begin{aligned}
& \langle \bar{q}_{v_1} \gamma_5 q_{v_2}(x) \bar{q}_{v_2} \gamma_5 q_{v_1}(0) \rangle_c = \\
& -\Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left[\frac{1}{x_1^{\text{eff}} + x_2^{\text{eff}}} \left(\frac{\Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\})}{\Sigma} + \frac{\Sigma_\nu^{\text{PQ}}(x_2^{\text{eff}}, \{z_i^{\text{eff}}\})}{\Sigma} \right) \right] \\
& + \frac{\Sigma^2}{2F^2} \left(\frac{\Delta \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} + \frac{\Delta \Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} + \frac{\nu^2}{x_1^2} + \frac{\nu^2}{x_2^2} \right. \\
& \left. - 2D_\nu^{\text{PQ}}(x_1, x_2, \{z_i\}) - \frac{2\nu^2}{x_1 x_2} \right) \bar{\Delta}(0|x) \\
& + \frac{\Sigma^2}{2F^2} \left[\frac{4}{x_1 - x_2} \left(\frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{\Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} \right) \right] \bar{G}(0, 0|x) \\
& - \frac{\Sigma^2}{2F^2} \sum_j^{N_f} \left[\frac{2}{x_1^2 - z_j^2} \left(\frac{x_1 \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right) \right. \\
& \left. + \frac{2}{x_2^2 - z_j^2} \left(\frac{x_2 \Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right) \right] \bar{\Delta}(M_{jv}^2|x).
\end{aligned} \tag{5.17}$$

We have here three types of ξ -correlators,

$$\bar{\Delta}(0|x) = \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{p^2}, \tag{5.18}$$

$$\bar{\Delta}(M_{iv}|x) = \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{p^2 + z_i/F^2 V}, \tag{5.19}$$

$$\bar{G}(0, 0|x) = \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx} (m_0^2 + \alpha p^2)/N_c}{p^4 \mathcal{F}(p^2)}. \tag{5.20}$$

Note that if the sea quarks are much smaller than the cut-off of ChPT, but still in the p -regime, one can take the $m_0 \rightarrow \infty$ limit. It leads to

$$\begin{aligned}
\bar{G}(0, 0|x) & \rightarrow \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{p^4 \left(\sum_i^{N_f} \frac{1}{p^2 + M_{ii}^2} \right)} \\
& = \frac{1}{N_f} \left(\bar{\Delta}(0|x) + \sum_i^{N_f} \frac{M_{ii}^2 e^{ipx}}{p^4} \right) + O(M_{ii}^4),
\end{aligned} \tag{5.21}$$

where the well-known double pole contribution appears due to a mismatch of the sea and valence quark masses. Further simplification is possible when all the sea quarks are in the ϵ -regime,

$$\bar{\Delta}(M_{iv}|x) \rightarrow \bar{\Delta}(0|x), \quad \bar{G}(0, 0|x) \rightarrow \frac{1}{N_f} \bar{\Delta}(0|x). \tag{5.22}$$

An interesting special case is the degenerate limit $x_1 = x_2$, where the above formulae become

$$\begin{aligned}
\langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_1} q_{v_1}(0) \rangle_d = & -\Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \frac{\Delta \Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\})}{\Sigma} \\
& - \frac{\Sigma^2}{F^2} \left(\frac{\partial_x \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{x_1 \Sigma} \right) \bar{\Delta}(0|x) \\
& + \frac{\Sigma^2}{F^2} \frac{2\nu^2}{x_1^2} \bar{G}(0, 0|x), \tag{5.23}
\end{aligned}$$

$$\begin{aligned}
\langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_1} \gamma_5 q_{v_1}(0) \rangle_d = & \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \frac{\nu^2}{(x_1^{\text{eff}})^2} \\
& - \frac{\Sigma^2}{F^2} \left(\frac{\partial_x \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{x_1 \Sigma} \right) \bar{\Delta}(0|x) \\
& - \frac{2\Sigma^2}{F^2} \frac{\Delta \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} \bar{G}(0, 0|x), \tag{5.24}
\end{aligned}$$

$$\begin{aligned}
\langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_1} q_{v_1}(0) \rangle_c = & \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\})}{\Sigma} - \frac{2\Sigma^2}{F^2} \frac{\nu^2}{x_1^2} \bar{\Delta}(0|x) \\
& - \frac{2\Sigma^2}{F^2} \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{x_1 \Sigma} \bar{G}(0, 0|x) \\
& + \frac{\Sigma^2}{2F^2} \sum_j^{N_f} \left[\frac{4}{x_1^2 - z_j^2} \left(\frac{x_1 \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right) \right] \bar{\Delta}(M_{jv}^2|x), \tag{5.25}
\end{aligned}$$

$$\begin{aligned}
\langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_1} \gamma_5 q_{v_1}(0) \rangle_c = & -\Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \frac{\Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\})}{x_1^{\text{eff}} \Sigma} + \frac{2\Sigma^2}{F^2} \frac{\Delta \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} \bar{\Delta}(0|x) \\
& + \frac{2\Sigma^2}{F^2} \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} \bar{G}(0, 0|x) \\
& - \frac{\Sigma^2}{2F^2} \sum_j^{N_f} \left[\frac{4}{x_1^2 - z_j^2} \left(\frac{x_1 \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right) \right] \bar{\Delta}(M_{jv}^2|x), \tag{5.26}
\end{aligned}$$

5.2 Singlet and flavored meson correlators

To compare with lattice QCD data, the case with arbitrary N_v -flavor valence quarks is interesting. In particular, N_v can be different from N_f , the number of sea quarks. In this paper we have aimed at the ϵ -regime predictions, and of course we always take the valence quarks to be in that regime. With the qualification mentioned in section 2, one can also be more general, and consider the sea quarks to be just marginally in the ϵ -regime, or even entirely in the p -regime. The finite volume is a $L^3 \times T$ box, where L and T are the spacial and temporal extents, respectively. The zero-momentum projections of singlet and $(N_v^2 - 1)$ -plet correlators are then obtained

$$\begin{aligned}
& \int d^3x \langle S^0(x) S^0(0) \rangle \\
& \equiv \int d^3x \left[N_v \langle \bar{q}_{v1} q_{v1}(x) \bar{q}_{v1} q_v(0) \rangle_c + N_v^2 \langle \bar{q}_{v1} q_{v1}(x) \bar{q}_{v1} q_{v1}(0) \rangle_d \right] \\
& = L^3 \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left[N_v \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\})}{\Sigma} - N_v^2 \frac{\Delta \Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\})}{\Sigma} \right] \\
& \quad - \frac{\Sigma^2}{2F^2} \left[\frac{4N_v \nu^2}{x_1^2} + 2N_v^2 \left(\frac{\partial_x \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{x_1 \Sigma} \right) \right] a(t/T) \\
& \quad - \frac{\Sigma^2}{2F^2} \left[4N_v \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{x_1 \Sigma} - 4N_v^2 \frac{\nu^2}{x_1^2} \right] b(t/T) \\
& \quad + \frac{N_v \Sigma^2}{2F^2} \sum_j^{N_f} \left[\frac{4}{x_1^2 - z_j^2} \left(\frac{x_1 \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right) \right] c_j(t/T),
\end{aligned} \tag{5.27}$$

$$\begin{aligned}
& \int d^3x \langle P^0(x) P^0(0) \rangle \\
& \equiv - \int d^3x \left[N_v \langle \bar{q}_{v1} \gamma_5 q_{v1}(x) \bar{q}_{v1} \gamma_5 q_{v1}(0) \rangle_c + N_v^2 \langle \bar{q}_{v1} \gamma_5 q_{v1}(x) \bar{q}_{v1} \gamma_5 q_{v1}(0) \rangle_d \right] \\
& = L^3 \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left[N_v \frac{\Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\})}{x_1^{\text{eff}} \Sigma} - N_v^2 \frac{\nu^2}{(x_1^{\text{eff}})^2} \right] \\
& \quad - \frac{\Sigma^2}{2F^2} \left[\frac{4N_v \Delta \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} \right. \\
& \quad \quad \left. - 2N_v^2 \left(\frac{\partial_x \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{x_1 \Sigma} \right) \right] a(t/T) \\
& \quad - \frac{\Sigma^2}{2F^2} \left[4N_v \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - 4N_v^2 \frac{\Delta \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} \right] b(t/T) \\
& \quad + \frac{N_v \Sigma^2}{2F^2} \sum_j^{N_f} \left[\frac{4}{x_1^2 - z_j^2} \left(\frac{x_1 \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right) \right] c_j(t/T),
\end{aligned} \tag{5.28}$$

$$\begin{aligned}
& \int d^3x \langle S^a(x) S^b(0) \rangle \equiv \frac{\delta_{ab}}{2} \int d^3x \langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_1} q_{v_1}(0) \rangle_c \\
& = \frac{\delta_{ab}}{2} \left[L^3 \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\})}{\Sigma} - \frac{2\Sigma^2}{F^2} \frac{\nu^2}{x_1^2} a(t/T) \right. \\
& \quad \left. - \frac{2\Sigma^2}{F^2} \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{x_1 \Sigma} b(t/T) \right. \\
& \quad \left. + \frac{\Sigma^2}{2F^2} \sum_j^{N_f} \left[\frac{4}{x_1^2 - z_j^2} \left(\frac{x_1 \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right) \right] c_j(t/T) \right], \tag{5.29}
\end{aligned}$$

$$\begin{aligned}
& \int d^3x \langle P^a(x) P^b(0) \rangle \equiv -\frac{\delta_{ab}}{2} \int d^3x \langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_1} \gamma_5 q_{v_1}(0) \rangle_c \\
& = -\frac{\delta_{ab}}{2} \left[-L^3 \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \frac{\Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\})}{x_1^{\text{eff}} \Sigma} + \frac{2\Sigma^2}{F^2} \frac{\Delta \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} a(t/T) \right. \\
& \quad \left. + \frac{2\Sigma^2}{F^2} \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} b(t/T) \right. \\
& \quad \left. - \frac{\Sigma^2}{2F^2} \sum_j^{N_f} \left[\frac{4}{x_1^2 - z_j^2} \left(\frac{x_1 \Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - \frac{z_j \Sigma_\nu^{\text{full}(N_f, j)}(\{z_i\})}{\Sigma} \right) \right] c_j(t/T) \right], \tag{5.30}
\end{aligned}$$

where $a(t/T)$, $b(t/T)$ and $c_j(t/T)$ are defined by

$$a(t/T) \equiv \int d^3x \bar{\Delta}(0|x) = \frac{T}{2} \left[\left(\frac{t}{T} - \frac{1}{2} \right)^2 - \frac{1}{12} \right], \tag{5.31}$$

$$\begin{aligned}
b(t/T) & \equiv \int d^3x \bar{G}(0, 0|x) = \int d^3x \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{p^4 \left(\sum_i^{N_f} \frac{1}{p^2 + M_{ii}^2} \right)} \\
& = \frac{1}{N_f} \frac{T}{2} \left[\left(\frac{t}{T} - \frac{1}{2} \right)^2 - \frac{1}{12} \right] \\
& \quad - \left(\sum_i^{N_f} \frac{M_{ii}^2}{N_f^2} \right) \frac{T^3}{24} \left[\left(\frac{t}{T} \right)^2 \left(\frac{t}{T} - 1 \right)^2 - \frac{1}{30} \right] + O(M_{ii}^4), \tag{5.32}
\end{aligned}$$

$$\begin{aligned}
c_j(t/T) & \equiv \int d^3x \bar{\Delta}(M_{jv}^2|x) = \frac{\cosh(M_{jv}(T/2 - t))}{2M_{jv} \sinh(M_{jv}T/2)} - \frac{1}{M_{jv}^2 T} \\
& = \frac{T}{2} \left[\left(\frac{t}{T} - \frac{1}{2} \right)^2 - \frac{1}{12} \right] + M_{jv}^2 \frac{T^3}{24} \left[\left(\frac{t}{T} \right)^2 \left(\frac{t}{T} - 1 \right)^2 - \frac{1}{30} \right] \\
& \quad + O(M_{jv}^4), \tag{5.33}
\end{aligned}$$

where $M_{jv}^2 = m_j \Sigma / F^2$ and $M_{ii}^2 = 2m_i \Sigma / F^2$.

In Fig. 2, Fig. 3 and Fig. 4, we plot, as examples, the flavored pseudo-scalar and scalar correlators (we simply denote Eq.(5.29) and Eq.(5.30) as $\langle S^a(t) \rangle$ and $\langle P^a(t) \rangle$ for $a = b$). We use $\Sigma = (250 \text{ MeV})^3$, $F = 93 \text{ MeV}$, $x_1^{\text{eff}}/x_1 = 1$ and $L = T = 2 \text{ fm}$ as inputs.

5.3 Ward-Takahashi identities

In this subsection, we check that the above results satisfy the Ward-Takahashi identities under the chiral rotation of the degenerate N_v valence quarks. For the singlet chiral rotation, one obtains

$$\langle (\partial_\mu A_\mu^0(x) - 2mP^0(x) - 2iN_v\omega(x)\mathcal{O}(0)) \rangle = -\langle \delta\mathcal{O}(0) \rangle \delta(x), \quad (5.34)$$

for any operator $\mathcal{O}(x)$, where

$$\begin{aligned} A_\mu^0(x) &= \sum_{v=1}^{N_v} \bar{q}_v(x) \gamma_\mu \gamma_5 q_v(x), \quad P^0(x) = \sum_{v=1}^{N_v} \bar{q}_v(x) \gamma_5 q_v(x), \\ \omega(x) &= \frac{1}{16\pi^2} \text{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu}(x), \end{aligned} \quad (5.35)$$

and $\delta\mathcal{O}$ denotes the chiral variation of \mathcal{O} . Note that Eq.(5.34) holds not only in θ -vacuum but also in a fixed topological sector. The identities for $\mathcal{O}(x) = P^0(x)$ and $\mathcal{O}(x) = \omega(x)$ and their integration over the volume give an equation

$$\int d^4x \langle P^0(x) P^0(0) \rangle = -\frac{N_v^2 \nu^2}{m^2 V} - \frac{\langle S^0(x) \rangle}{m} = -\frac{N_v^2 \nu^2}{m^2 V} + \frac{N_v \Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\}) x_1^{\text{eff}}}{m x_1}, \quad (5.36)$$

where $S^0(x) = \sum_{v=1}^{N_v} \bar{q}_v(x) q_v(x)$, which coincides with Eq.(5.28). Note here we used

$$\int dt a(t/T) = \int dt b(t/T) = \int dt c_j(t/T) = 0. \quad (5.37)$$

In the same way, the flavored identity

$$\langle (\partial_\mu A_\mu^a(x) - 2mP^a(x)) \mathcal{O}(0) \rangle = -\langle \delta^a \mathcal{O}(0) \rangle \delta(x), \quad (5.38)$$

for $\mathcal{O}(x) = P^a(x)$ and $\mathcal{O}(x) = \partial_\mu A_\mu^a(x)$ gives

$$\int d^4x \langle P^a(x) P^a(0) \rangle = -\frac{\langle S^0(x) \rangle}{2N_v m} = \frac{\Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\}) x_1^{\text{eff}}}{2m x_1}, \quad (5.39)$$

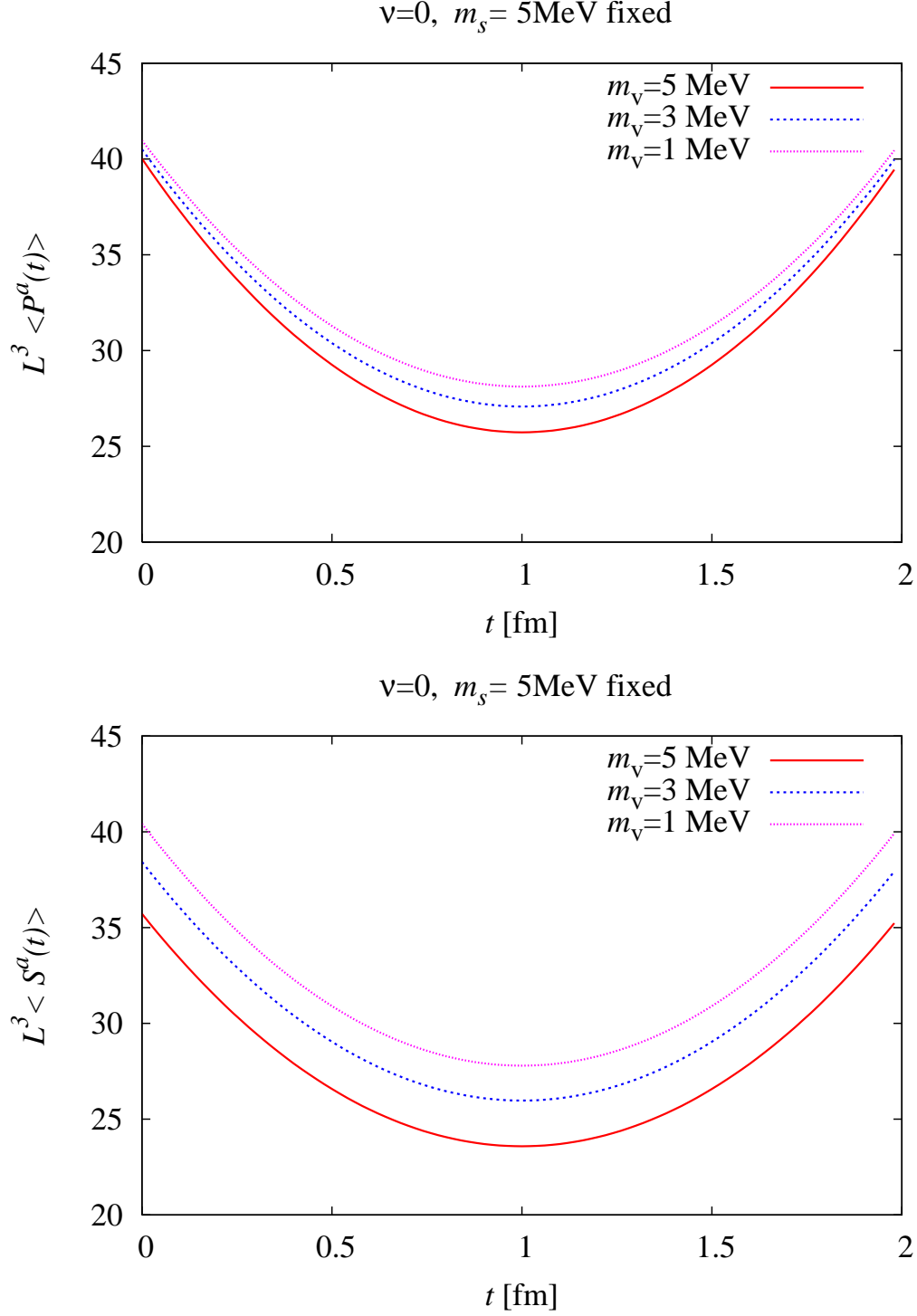


Fig. 2. Pseudo-scalar (top) and scalar (bottom) flavored correlators with both valence and sea quarks in the ϵ -regime, here for $\nu = 0$. The sea quark masses are fixed to 5 MeV. We use $\Sigma=(250 \text{ MeV})^3$, $F = 93 \text{ MeV}$, $x_1^{\text{eff}}/x_1 = 1$ and $L = T = 2 \text{ fm}$ as inputs.

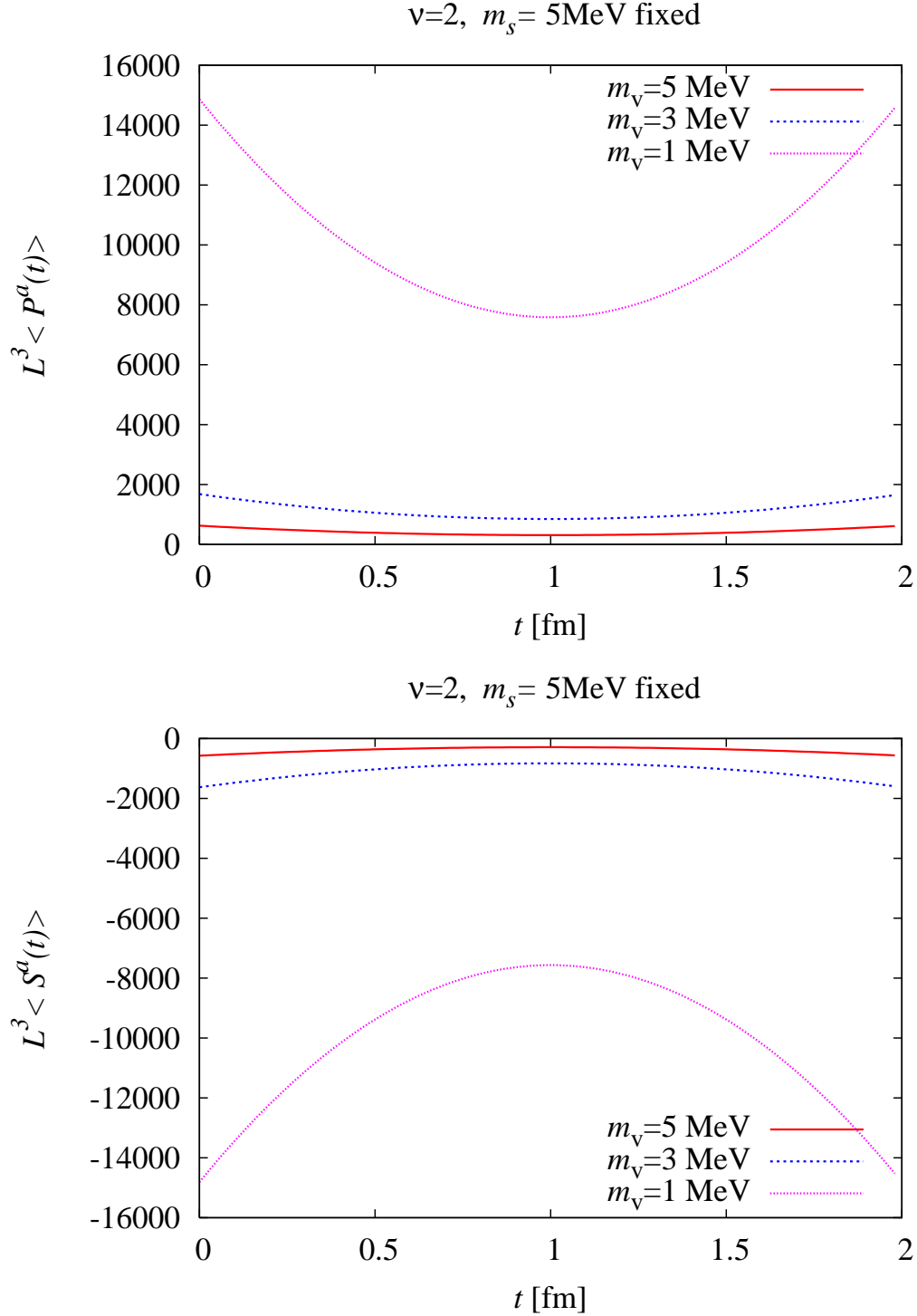


Fig. 3. The same as Fig. 2 but with $\nu = 2$. Note that the pseudo-scalar and scalar correlation functions almost sum up to zero.

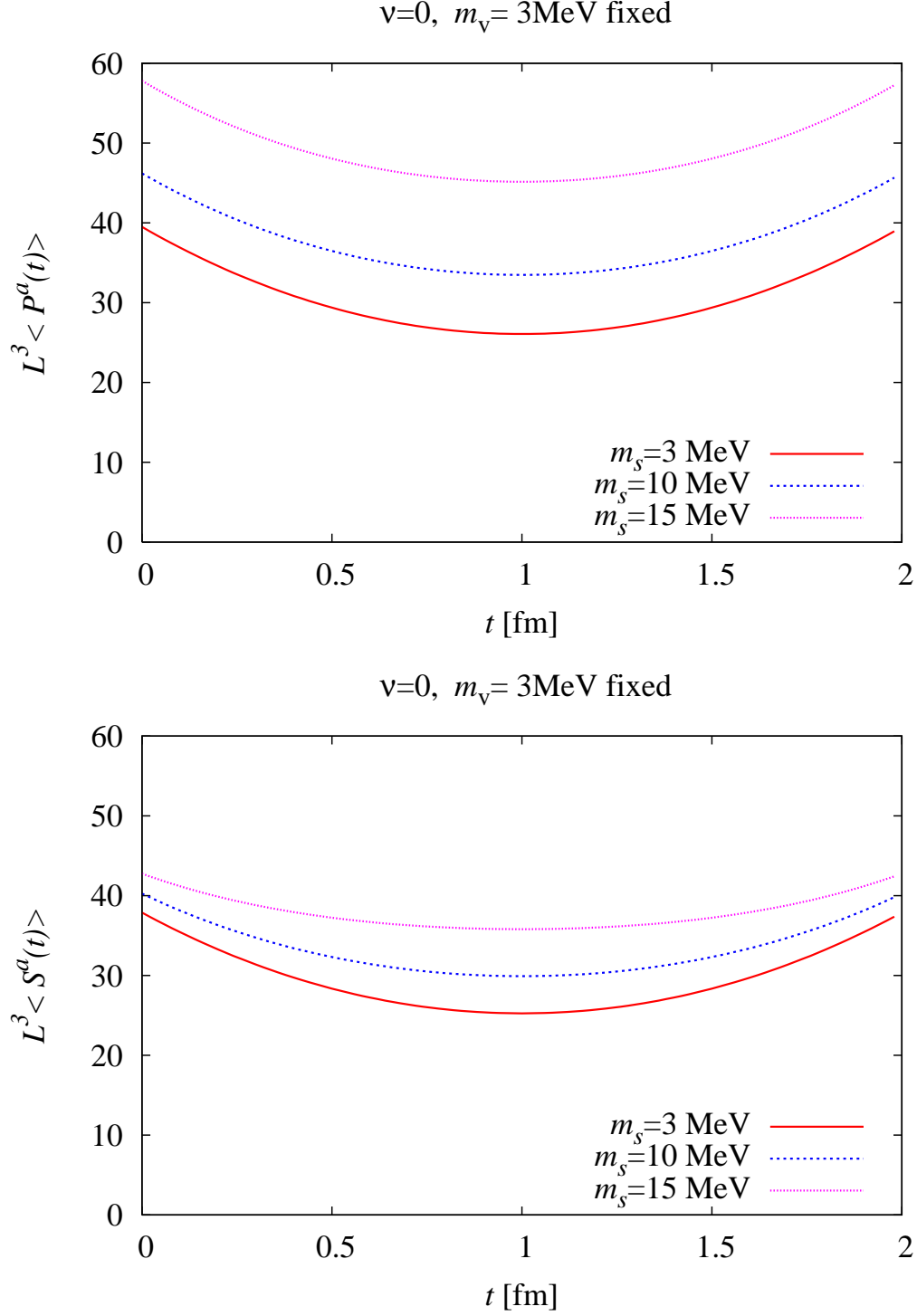


Fig. 4. The sea quark mass dependence of pseudo-scalar (top) and scalar (bottom) flavored correlators with the valence quark mass fixed at 3 MeV. The parameters are chosen the same as for Fig.2 and Fig.3. Here we assume that we can still ignore the m_0 and α terms for sea quark masses up to ~ 15 MeV.

which is also consistent with Eq.(5.30). Here the operator with the superscript “ a ” has the form of $\bar{q}_v(x)\gamma T^a q_v(x)$ with some gamma matrix γ , as in the conventional notation, where T^a denotes the a -th generator of $SU(N_v)$ group.

One can also confirm the similar identity [33]

$$\int d^4x \langle S^a(x) S^a(0) \rangle = \frac{1}{2N_v V} \frac{\partial}{\partial m_v} \langle S^0(x) \rangle = \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \frac{\partial}{2\partial m_{v_1}} \Sigma_\nu^{\text{PQ}}(x_1^{\text{eff}}, \{z_i^{\text{eff}}\}), \quad (5.40)$$

is consistent with Eq.(5.29). Since the asymptotic form of the partially quenched condensate in the chiral limit is known [7]

$$\frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} \sim \frac{|\nu|}{x_1} + \mathcal{O}(x_1), \quad (5.41)$$

it is not difficult to see that the known “quenched” identities for the $N_v = 2$ case [33],

$$\int d^4x [\langle P^0(x) P^0(0) \rangle - 4 \langle S^a(x) S^a(0) \rangle] \sim \frac{4|\nu|}{m_v^2 V} - \frac{4\nu^2}{m_v^2 V}, \quad (5.42)$$

$$\int d^4x [\langle P^a(x) P^a(0) \rangle - 4 \langle S^a(x) S^a(0) \rangle] \sim \frac{4|\nu|}{m_v^2 V}, \quad (5.43)$$

also hold in the limit $m_v \rightarrow 0$ in this partially quenched theory.

5.4 Quenched and full degenerate N_f flavor limits

In this subsection, we show how to reproduce the known results of both the fully quenched theory and the unquenched theory by taking the $z_i \rightarrow \infty$ limit and the $x = z_1 = z_2 = \dots = z$ limits, respectively. The valence quarks are chosen to be degenerate in the ϵ -regime.

First consider the quenched limit of connected correlators Eq.(5.25) and Eq.(5.26),

$$\begin{aligned} \langle S^a(x) S^a(0) \rangle^{\text{que}} &= \lim_{z_i \rightarrow \infty} \frac{1}{2} \langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_1} q_{v_1}(0) \rangle_c \\ &= \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \frac{\partial_x \Sigma_\nu^{\text{que}}(x_1^{\text{eff}})}{2\Sigma} \\ &\quad + \frac{\Sigma^2}{2F^2} \left[-\frac{2\nu^2}{x_1^2} \bar{\Delta}(0|x) - \frac{2\Sigma_\nu^{\text{que}}(x_1)}{x_1 \Sigma} \bar{G}(0, 0|x) \right], \end{aligned} \quad (5.44)$$

$$\begin{aligned}
\langle P^a(x)P^a(0) \rangle^{que} &= \lim_{z_i \rightarrow \infty} -\frac{1}{2} \langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_1} \gamma_5 q_{v_1}(0) \rangle_c \\
&= \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \frac{\Sigma_\nu^{que}(x_1^{\text{eff}})}{2x_1^{\text{eff}}\Sigma} \\
&\quad - \frac{\Sigma^2}{2F^2} \left[-2 \left(1 + \frac{\nu^2}{x_1^2} \right) \bar{\Delta}(0|x) + \frac{2\partial_x \Sigma_\nu^{que}(x_1)}{\Sigma} \bar{G}(0,0|x) \right].
\end{aligned} \tag{5.45}$$

Noting

$$\bar{\Delta}(0|x) = \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{p^2}, \quad \bar{G}(0,0|x) = \frac{1}{V} \sum_{p \neq 0} \frac{1}{N_c} \left(\frac{e^{ipx} m_0^2}{p^4} + \frac{e^{ipx} \alpha}{p^2} \right), \tag{5.46}$$

in the quenched limit, one can see that Eq.(5.44) and Eq.(5.45) agree with the quenched results in [5].

Next we construct $N_v = 1$ singlet correlation functions in the quenched limit,

$$\begin{aligned}
\langle S^0(x)S^0(0) \rangle^{que} &= \lim_{z_i \rightarrow \infty} (\langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_1} q_{v_1}(0) \rangle_c + \langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_1} q_{v_1}(0) \rangle_d) \\
&= \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left[\frac{\partial_x \Sigma_\nu^{que}(x_1^{\text{eff}})}{\Sigma} + 1 + \frac{\nu^2}{(x_1^{\text{eff}})^2} \right] \\
&\quad + \frac{\Sigma^2}{2F^2} \left[\left(-\frac{4\nu^2}{x_1^2} + \frac{2\Sigma_\nu^{que}(x_1)}{x_1 \Sigma} - \frac{2\partial_x \Sigma_\nu^{que}(x_1)}{\Sigma} \right) \bar{\Delta}(0|x) \right. \\
&\quad \left. - \left(\frac{4\Sigma_\nu^{que}(x_1)}{x_1 \Sigma} - \frac{4\nu^2}{x_1^2} \right) \bar{G}(0,0|x) \right],
\end{aligned} \tag{5.47}$$

$$\begin{aligned}
\langle P^0(x)P^0(0) \rangle^{que} &= \lim_{z_i \rightarrow \infty} (\langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_1} \gamma_5 q_{v_1}(0) \rangle_c - \langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_1} \gamma_5 q_{v_1}(0) \rangle_d) \\
&= \Sigma^2 \left(\frac{x_1^{\text{eff}}}{x_1} \right)^2 \left(\frac{\Sigma_\nu^{que}(x_1^{\text{eff}})}{x_1^{\text{eff}} \Sigma} - \frac{\nu^2}{(x_1^{\text{eff}})^2} \right) \\
&\quad - \frac{\Sigma^2}{2F^2} \left[-\left(4 + \frac{4\nu^2}{x_1^2} - \frac{2\Sigma_\nu^{que}(x_1)}{x_1 \Sigma} + \frac{2\partial_x \Sigma_\nu^{que}(x_1)}{\Sigma} \right) \bar{\Delta}(0|x) \right. \\
&\quad \left. + \left(\frac{4\partial_x \Sigma_\nu^{que}(x_1)}{\Sigma} + 4 + \frac{4\nu^2}{x_1^2} \right) \bar{G}(0,0|x) \right],
\end{aligned} \tag{5.48}$$

which are also equivalent to the results of [5].

The full degenerate N_f -flavor limit ($x \rightarrow z = z_1 = z_2 = \dots$) of the connected correlators in the ϵ -regime are

$$\begin{aligned}
\langle S^a(x)S^a(0) \rangle^{full} &= \frac{1}{2} \langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_1} q_{v_1}(0) \rangle_c |_{x=z_1=z_2=\dots=z} \\
&= \Sigma^2 \left(\frac{z^{\text{eff}}}{z} \right)^2 \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z^{\text{eff}}, z^{\text{eff}}, \dots\})}{2\Sigma} \Big|_{x=z^{\text{eff}}} \\
&\quad - \frac{\Sigma^2}{2F^2} \left[\frac{2\nu^2}{z^2} + \frac{2\Sigma_\nu^{N_f}(z)}{N_f z \Sigma} - N_f \left(\frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z, z, \dots\})}{\Sigma} \right) \Big|_{x=z} \right. \\
&\quad \left. + \frac{\Sigma_\nu^{N_f}(z)}{z \Sigma} \right] \bar{\Delta}(0|x),
\end{aligned} \tag{5.49}$$

$$\begin{aligned}
\langle P^a(x)P^a(0) \rangle^{full} &= -\frac{1}{2} \langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_1} \gamma_5 q_{v_1}(0) \rangle_c |_{x=z_1=z_2=\dots=z} \\
&= \Sigma^2 \left(\frac{z^{\text{eff}}}{z} \right)^2 \frac{\Sigma_\nu^{N_f}(z^{\text{eff}})}{2z^{\text{eff}} \Sigma} \\
&\quad + \frac{\Sigma^2}{2F^2} \left[-\frac{2\Delta \Sigma_\nu^{\text{PQ}}(z, \{z, z, \dots\})}{\Sigma} - \frac{2\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z, z, \dots\})}{N_f \Sigma} \right] \Big|_{x=z} \\
&\quad + N_f \left(\frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z, z, \dots\})}{\Sigma} \Big|_{x=z} + \frac{\Sigma_\nu^{N_f}(z)}{z \Sigma} \right) \bar{\Delta}(0|x).
\end{aligned} \tag{5.50}$$

Note here that we set $M_{vv}^2 = M_{vv}^2 = 0$, $\bar{G}(0, 0|x) = \bar{\Delta}(0|x)/N_f$, $\Sigma_\nu^{N_f}(z)/\Sigma$ is the full degenerate N_f -flavor condensate defined by Eq.(4.35), and z^{eff} is given by Eq.(3.10). To eliminate the partially quenched expression, $\Sigma^{\text{PQ}}/\Sigma$, we use Eq.(4.34), Eq.(4.38) and Eq.(4.39) to obtain

$$\begin{aligned}
\langle S^a(x)S^a(0) \rangle^{full} &= \frac{N_f \Sigma^2}{2(N_f^2 - 1)} \left(\frac{z^{\text{eff}}}{z} \right)^2 \left[-\left(\frac{\Sigma_\nu^{N_f}(z^{\text{eff}})}{\Sigma} \right)^2 \right. \\
&\quad \left. - \frac{1}{N_f} \left(\frac{\Sigma_\nu^{N_f}(z^{\text{eff}})}{\Sigma} \right)' - N_f \frac{\Sigma_\nu^{N_f}(z^{\text{eff}})}{z^{\text{eff}} \Sigma} + 1 + \frac{\nu^2}{(z^{\text{eff}})^2} \right] \\
&\quad - \frac{\Sigma^2}{2(N_f^2 - 1)F^2} \left[(N_f^2 - 2) \frac{\nu^2}{z^2} + \frac{(3N_f^2 - 2) \Sigma_\nu^{N_f}(z)}{N_f z \Sigma} \right. \\
&\quad \left. + N_f^2 \left(\left(\frac{\Sigma_\nu^{N_f}(z)}{\Sigma} \right)^2 + \frac{1}{N_f} \left(\frac{\Sigma_\nu^{N_f}(z)}{\Sigma} \right)' - 1 \right) \right] \bar{\Delta}(0|x),
\end{aligned} \tag{5.51}$$

$$\begin{aligned}
\langle P^a(x)P^a(0) \rangle^{full} &= \Sigma^2 \left(\frac{z^{\text{eff}}}{z} \right)^2 \frac{\Sigma_\nu^{N_f}(z^{\text{eff}})}{2z^{\text{eff}}\Sigma} \\
&+ \frac{\Sigma^2}{2(N_f^2 - 1)F^2} \left[(N_f^2 - 4) \frac{\nu^2}{z^2} + 3N_f \frac{\Sigma_\nu^{N_f}(z)}{z\Sigma} + N_f^2 - 4 \right. \\
&\left. + (N_f^2 + 2) \left(\left(\frac{\Sigma_\nu^{N_f}(z)}{\Sigma} \right)^2 + \frac{1}{N_f} \left(\frac{\Sigma_\nu^{N_f}(z)}{\Sigma} \right)' \right) \right] \bar{\Delta}(0|x), \quad (5.52)
\end{aligned}$$

which agree with those in the full N_f -flavor theory.

In the same way, one can see that the singlet correlators

$$\begin{aligned}
\langle S^0(x)S^0(0) \rangle^{full} &= N_f \langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_1} q_{v_1}(0) \rangle_c + N_f^2 \langle \bar{q}_{v_1} q_{v_1}(x) \bar{q}_{v_1} q_{v_1}(0) \rangle_d \\
&= \Sigma^2 \left(\frac{z^{\text{eff}}}{z} \right)^2 \left[N_f \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z^{\text{eff}}, z^{\text{eff}}, \dots\})}{\Sigma} \right]_{x=z^{\text{eff}}} \\
&\quad - N_f^2 \frac{\Delta \Sigma_\nu^{\text{PQ}}(z^{\text{eff}}, \{z^{\text{eff}}, z^{\text{eff}}, \dots\})}{\Sigma} \Big] \\
&\quad + \frac{\Sigma^2}{2F^2} \left[4(N_f^2 - 1) \frac{\Sigma_\nu^{N_f}(z)}{z\Sigma} \right] \bar{\Delta}(0|x), \quad (5.53)
\end{aligned}$$

$$\begin{aligned}
\langle P^0(x)P^0(0) \rangle^{full} &= -N_f \langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_1} \gamma_5 q_{v_1}(0) \rangle_c - N_f^2 \langle \bar{q}_{v_1} \gamma_5 q_{v_1}(x) \bar{q}_{v_1} \gamma_5 q_{v_1}(0) \rangle_d \\
&= \Sigma^2 \left(\frac{z^{\text{eff}}}{z} \right)^2 \left[N_f \frac{\Sigma_\nu^{N_f}(z^{\text{eff}})}{z^{\text{eff}}\Sigma} - N_f^2 \frac{\nu^2}{(z^{\text{eff}})^2} \right] \\
&\quad + \frac{\Sigma^2}{2F^2} \left[4(N_f^2 - 1) \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z, z, \dots\})}{\Sigma} \right]_{x=z} \bar{\Delta}(0|x), \quad (5.54)
\end{aligned}$$

are also consistent with the known expressions in [5].

6 Conclusions

In this paper, we have discussed partially quenched chiral perturbation theory (PQChPT) in the ϵ -regime, and in the mixed ϵ and p -regime.

Using the 1-loop improved chiral condensate and its derivative as building blocks, we have calculated various zero-mode group integrals in the replica

limit. These integrals are necessary for the computation of mesonic correlation functions in the partially quenched theory. We have also derived a non-trivial identity which is a consequence of unitarity of the graded or replicated group.

With these zero-mode integrals and the Feynman rules for the non-zero modes, we have calculated the mesonic correlation functions for both connected and disconnected pseudo-scalar and scalar channels with non-degenerate quark masses, both of the valence and sea kind. Among others, our results can be applied to the mesons that consist of two non-degenerate valence quarks. For a demonstration, we have plotted the flavored pseudo-scalar and scalar correlators with a realistic choice of input parameters. As expected, they show a non-trivial valence (sea) quark mass dependence with a fixed sea (valence) quark mass.

These meson correlators were shown to have the correct quenched and degenerate full N_f -flavor theory limits. We have not addressed the implicit flavor dependence of Σ , or F , the fundamental parameters in the infinite volume limit. In order to complete the smooth connection among the theories with different number of flavors, one has to match the value of them [28,29,30,31,32]. The flavor dependence is expected to be rather weak, but the matching is interesting and important for the future works.

Our results are useful for the analysis of unquenched lattice QCD simulations in many ways. The various valence quark masses can be used for each set of fixed sea quark masses. Even if the physical pions are just barely in the ϵ -regime, one can put the valence pions very safely in the ϵ -regime and compare numerical data with our formulae for the partially quenched correlation functions.

Acknowledgments

We would like to thank J.J.M.Verbaarschot and K.Splittorff for fruitful discussions. HF wish to thank the members of Niels Bohr Institute for warm hospitality during his stay. The work of HF was supported by a Grant-in-Aid of the Japanese Ministry of Education (No.18840045), and the work of PHD was supported in part by the European Community Network ENRAGE (MRTN-CT-2004-005616).

A Summary of group integrals

Here we summarize the group integrals in the replica limit, which are necessary for the meson correlators. See Sec.4.1 for the details. The formulae for one valence index are,

$$\frac{1}{2}\langle(U_{vv} + U_{vv}^\dagger)\rangle = \frac{\Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma}, \quad (\text{A.1})$$

$$\frac{1}{4}\langle(U_{vv} + U_{vv}^\dagger)^2\rangle = \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} - \frac{\Delta \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma}, \quad (\text{A.2})$$

$$\frac{1}{2}\langle(U_{vv} - U_{vv}^\dagger)\rangle = -\frac{\nu}{x}, \quad (\text{A.3})$$

$$\frac{1}{4}\langle(U_{vv} - U_{vv}^\dagger)^2\rangle = -\frac{\Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{x\Sigma} + \frac{\nu^2}{x^2}, \quad (\text{A.4})$$

$$\begin{aligned} \langle U_{vv} U_{vv}^\dagger \rangle &= \frac{1}{4}\langle(U_{vv} + U_{vv}^\dagger)^2\rangle - \frac{1}{4}\langle(U_{vv} - U_{vv}^\dagger)^2\rangle \\ &= \frac{\partial_x \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} - \frac{\Delta \Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{\Sigma} + \frac{\Sigma_\nu^{\text{PQ}}(x, \{z_i\})}{x\Sigma} - \frac{\nu^2}{x^2}. \end{aligned} \quad (\text{A.5})$$

For two valence indices,

$$\frac{1}{4}\langle(U_{v_1 v_1} + U_{v_1 v_1}^\dagger)(U_{v_2 v_2} + U_{v_2 v_2}^\dagger)\rangle = D_\nu^{\text{PQ}}(x_1, x_2, \{z_i\}), \quad (\text{A.6})$$

$$\frac{1}{4}\langle(U_{v_1 v_1} - U_{v_1 v_1}^\dagger)(U_{v_2 v_2} - U_{v_2 v_2}^\dagger)\rangle = \frac{\nu^2}{x_1 x_2}, \quad (\text{A.7})$$

$$\langle U_{v_1 v_1} U_{v_2 v_2} \rangle + \langle U_{v_1 v_1}^\dagger U_{v_2 v_2}^\dagger \rangle = 2D_\nu^{\text{PQ}}(x_1, x_2, \{z_i\}) + \frac{2\nu^2}{x_1 x_2}, \quad (\text{A.8})$$

where D_ν^{PQ} is defined in Eq.(4.13). Similarly,

$$\begin{aligned} \frac{1}{4}\langle(U_{v_1 v_2} \pm U_{v_2 v_1}^\dagger)^2\rangle &= \frac{1}{4}\langle(U_{v_2 v_1} \pm U_{v_1 v_2}^\dagger)^2\rangle \\ &= \frac{\pm 1}{2}\langle U_{v_1 v_2} U_{v_2 v_1}^\dagger \rangle = \frac{\pm 1}{2}\langle U_{v_2 v_1} U_{v_1 v_2}^\dagger \rangle \\ &= \frac{\pm 1}{x_1^2 - x_2^2} \left(x_1 \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - x_2 \frac{\Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} \right), \end{aligned} \quad (\text{A.9})$$

$$\frac{1}{4}\langle U_{v_1 v_2}^2 + (U_{v_2 v_1}^\dagger)^2 \rangle = 0, \quad (\text{A.10})$$

$$\frac{1}{4} \langle (U_{v_1 v_2} \pm U_{v_2 v_1}^\dagger) (U_{v_2 v_1} \pm U_{v_1 v_2}^\dagger) \rangle = \frac{1}{x_1^2 - x_2^2} \left(x_2 \frac{\Sigma_\nu^{\text{PQ}}(x_1, \{z_i\})}{\Sigma} - x_1 \frac{\Sigma_\nu^{\text{PQ}}(x_2, \{z_i\})}{\Sigma} \right), \quad (\text{A.11})$$

are obtained.

References

- [1] J. Gasser and H. Leutwyler, Phys. Lett. B **188**, 477 (1987).
- [2] H. Neuberger, Phys. Rev. Lett. **60** (1988) 889.
- [3] F. C. Hansen, Nucl. Phys. B **345**, 685 (1990); F. C. Hansen and H. Leutwyler, Nucl. Phys. B **350**, 201 (1991).
- [4] P. Hasenfratz and H. Leutwyler, Nucl. Phys. B **343**, 241 (1990).
- [5] P. H. Damgaard, M. C. Diamantini, P. Hernandez and K. Jansen, Nucl. Phys. B **629**, 445 (2002) [arXiv:hep-lat/0112016].
- [6] P. H. Damgaard, P. Hernandez, K. Jansen, M. Laine and L. Lellouch, Nucl. Phys. B **656**, 226 (2003) [arXiv:hep-lat/0211020].
- [7] P. H. Damgaard and K. Splittorff, Nucl. Phys. B **572**, 478 (2000) [arXiv:hep-th/9912146].
- [8] P. H. Damgaard, Phys. Lett. B **476**, 465 (2000) [arXiv:hep-lat/0001002].
- [9] P. H. Damgaard, Nucl. Phys. B **608**, 162 (2001) [arXiv:hep-lat/0105010].
- [10] L. Giusti, C. Hoelbling, M. Luscher and H. Wittig, Comput. Phys. Commun. **153**, 31 (2003) [arXiv:hep-lat/0212012].
- [11] W. Bietenholz, T. Chiarappa, K. Jansen, K. I. Nagai and S. Shcheredin, JHEP **0402**, 023 (2004) [arXiv:hep-lat/0311012].
- [12] L. Giusti, P. Hernandez, M. Laine, P. Weisz and H. Wittig, JHEP **0401**, 003 (2004) [arXiv:hep-lat/0312012].
- [13] L. Giusti, P. Hernandez, M. Laine, P. Weisz and H. Wittig, JHEP **0404**, 013 (2004) [arXiv:hep-lat/0402002].
- [14] K. Ogawa and S. Hashimoto, Prog. Theor. Phys. **114**, 609 (2005) [arXiv:hep-lat/0505017].
- [15] H. Fukaya, S. Hashimoto and K. Ogawa, Prog. Theor. Phys. **114**, 451 (2005) [arXiv:hep-lat/0504018].
- [16] W. Bietenholz and S. Shcheredin, Nucl. Phys. B **754**, 17 (2006) [arXiv:hep-lat/0605013].

- [17] F. Bernardoni, P. H. Damgaard, H. Fukaya and P. Hernandez, *in preparation*.
- [18] C. W. Bernard and M. F. L. Golterman, Phys. Rev. D **49** (1994) 486 [arXiv:hep-lat/9306005].
- [19] S. R. Sharpe and N. Shoresh, Phys. Rev. D **64**, 114510 (2001) [arXiv:hep-lat/0108003].
- [20] H. Fukaya *et al.* [JLQCD Collaboration], PoS **LAT2006**, 050 (2006) [arXiv:hep-lat/0610024].
- [21] H. Fukaya *et al.* [JLQCD Collaboration], Phys. Rev. Lett. **98**, 172001 (2007) [arXiv:hep-lat/0702003].
- [22] H. Fukaya *et al.*, arXiv:0705.3322 [hep-lat].
- [23] K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. Lett. **90**, 041601 (2003) [arXiv:cond-mat/0209594].
- [24] Y. V. Fyodorov and G. Akemann, JETP Lett. **77** (2003) 438 [Pisma Zh. Eksp. Teor. Fiz. **77** (2003) 513] [arXiv:cond-mat/0210647].
- [25] F. Bernardoni and P. Hernandez, arXiv:0707.3887 [hep-lat].
- [26] P. H. Damgaard and K. Splittorff, Phys. Rev. D **62**, 054509 (2000) [arXiv:hep-lat/0003017].
- [27] J. C. Osborn, D. Toublan and J. J. M. Verbaarschot, Nucl. Phys. B **540** (1999) 317 [arXiv:hep-th/9806110].
- [28] L. Giusti, P. Hernandez, M. Laine, P. Weisz and H. Wittig, JHEP **0411**, 016 (2004) [arXiv:hep-lat/0407007].
- [29] P. Hernandez and M. Laine, JHEP **0610**, 069 (2006) [arXiv:hep-lat/0607027].
- [30] T. DeGrand, R. Hoffmann, S. Schaefer and Z. Liu, Phys. Rev. D **74**, 054501 (2006) [arXiv:hep-th/0605147].
- [31] T. DeGrand, Z. Liu and S. Schaefer, Phys. Rev. D **74**, 094504 (2006) [Erratum-ibid. D **74**, 099904 (2006)] [arXiv:hep-lat/0608019].
- [32] P. Hasenfratz, D. Hierl, V. Maillart, F. Niedermayer, A. Schafer, C. Weiermann and M. Weingart, arXiv:0707.0071 [hep-lat].
- [33] R. G. Edwards, U. M. Heller and R. Narayanan, Phys. Rev. D **59**, 094510 (1999) [arXiv:hep-lat/9811030].